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Monopoles, vortices, domain walls and D-branes: the rules of interaction

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Abstract: Non-abelian gauge theories in the Higgs phase admit a startling variety of BPS solitons. These include domain walls, vortex strings, confined monopoles threaded on vortex strings, vortex strings ending on domain walls, monopoles threaded on strings ending on domain walls, and more. After presenting a self-contained review of these objects, including several new results on the dynamics of domain walls, we go on to examine the possible interactions of solitons of various types. We point out the existence of a classical binding energy when the string ends on the domain wall which can be thought of as a BPS boojum with negative mass. We present an index theorem for domain walls in non-abelian gauge theories. We also answer questions such as: Which strings can end on which walls? What happens when monopoles pass through domain walls? What happens when domain walls pass through each other?

Keywords: Solitons Monopoles and Instantons, D-branes.

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Contents

1. Introduction

Non-abelian gauge theories in the phase where the gauge group is fully broken enjoy a wonderfully rich spectrum of solitons. In this paper we review some of the recent results on these objects and specify the allowed rules of interaction between the solitons of different types.

In section 2 we describe the theory in question: it is a $d = 3+1$ dimensional $\mathrm{U}(N_c)$ gauge theory with N_f Higgs fields in the fundamental representation and a single scalar field in the adjoint representation. These fields comprise the bosonic sector of $\mathcal{N} = 2$ supersymmetric

QCD and all the solitons described in this paper are BPS in nature, preserving some fraction of the supersymmetry. After examining the symmetry and vacuum structure of the theory, the remaining sections are devoted to studying each of the various solitons in turn. They are:

Domain walls: as we shall explain in section 2, the theory has a large number of isolated vacuum states. In fact, this number grows exponentially as the number of flavours is increased. This allows for a vast selection of co-dimension one domain wall solutions interpolating between different vacua and raises various questions: what is the correct way to classify the domain walls? Given two vacua, how many domain walls can interpolate between them? What properties do these domain walls have? In section 3 we shall review what is known about these domain walls and present answers to all these questions. An appendix contains an index theorem in which we determine the number of parameters of a given domain wall system.

Vortices: the theory also admits co-dimension two solitonic objects. These are vortex strings. In fact, in any given vacuum there are N_c different types of vortex strings corresponding to the magnetic field sitting in a different diagonal component of the adjoint-valued field strength. In section 4 we review recent work on the interactions of these different vortex strings.

Confined monopoles: famously, non-abelian gauge theories which are broken to U(1) factors admit 't Hooft-Polyakov monopoles. Recently it was realised that stable monopoles continue to exist in the theory even when the gauge group is broken completely. Moreover, these monopoles remain BPS. Rather than spreading out radially, the magnetic flux leaves the monopole in two collimated tubes which are identified with the vortex strings. From the perspective of the vortex, the monopole looks like a kink solution in the string. We review this object in section 5.

D-branes: the domain walls of our theory have an interesting property: a U(1) gauge field lives on their $d = 2 + 1$ dimensional worldvolume. Moreover, the vortex string can terminate on the domain wall where its end is electrically charged under the U(1) gauge field. In other words, the theory contains a semi-classical D-brane configuration. We describe this soliton in section 6 and show that there exists a property of this system missed in previous studies: a finite, classical binding energy between the end of the string and the domain wall. This binding energy can be thought of as a BPS, negative mass, magnetic monopole - or boojum - embedded within the domain wall.

Interactions: given these basic objects, it is natural to study the dynamics of the various components. For a single class of solitons — such as domain walls, or vortices — this is a well developed subject. But less attention has been paid to the dynamics of composite configurations that include multiple types of solitons. In sections 6 and 7, we describe the basic rules of classical interaction between solitons of different types. We answer questions such as: Which type of vortex can end on which type of domain

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wall? Can monopoles pass through domain walls? How do monopoles interact with the string binding energy? What becomes of strings and monopoles when domain walls pass through each other?

2. The theory

Throughout this paper we shall consider a class of $d = 3+1$ dimensional[1] non-abelian gauge theories with $\mathcal{N} = 2$ supersymmetry. We take a $\mathrm{U}(N_c)$ gauge group, coupled to N_f hypermultiplets transforming in the fundamental representation. Although the solitons exist most naturally in this supersymmetric context, for the most part we need only discuss the bosonic fields and we focus on these, commenting on the fermionic zero modes only in passing.

The $\mathrm{U}(N_c)$ vector multiplet includes the gauge field A_μ and a complex adjoint scalar field ϕ, together with fermions. The bosonic content of each hypermultiplet consists of two complex scalar fields q_i and $\tilde{q}_i$, where $i = 1, \ldots, N_f$ is the flavour index. Each field q transforms in the fundamental $\mathbf{N_c}$ representation of the gauge group, while $\tilde{q}$ transforms in the anti-fundamental $\bar{\mathbf{N}}_\mathbf{c}$ representation. Denoting spacetime indices by $M, N = 0, 1, 2, 3$, we obtain the bosonic part of the lagrangian[2]

$$\begin{aligned}
-\mathcal{L} = \frac{1}{2e^2}\,\mathrm{Tr}\left(\frac{1}{2}F_{MN}F^{MN} + |\mathcal{D}_M\phi|^2 + |[\phi,\phi^\dagger]|^2\right) + \sum_{i=1}^{N_f}\left(|\mathcal{D}_M q_i|^2 + |\mathcal{D}_M\tilde{q}_i|^2\right) + \\
+\,\mathrm{Tr}\left(\frac{e^2}{2}\left(\sum_{i=1}^{N_f}(q_i q_i^\dagger - \tilde{q}_i^\dagger\tilde{q}_i) - v^2\right)^2 + e^2\left|\sum_{i=1}^{N_f}\tilde{q}_i q_i\right|^2\right) + \\
+\frac{1}{2}\sum_{i=1}^{N_f}\left(q_i^\dagger\left\{\phi - m_i, \phi^\dagger - m_i^\dagger\right\}q_i + \tilde{q}_i\left\{\phi - m_i, \phi^\dagger - m_i^\dagger\right\}\tilde{q}_i^\dagger\right). \qquad (2.1)
\end{aligned}$$

Here the adjoint covariant derivative is given by $\mathcal{D}_M\phi = \partial_M\phi - i[A_M, \phi]$ while the fundamental covariant derivative is $\mathcal{D}_M q = \partial_M q - iA_M q$. The field strength is defined by $F_{MN} = i[\mathcal{D}_M, \mathcal{D}_N]$.

The theory contains three types of parameters: the gauge coupling e^2, the Higgs vacuum expectation value (vev) (or, in the language of supersymmetry, the Fayet-Iliopoulos (FI) parameter) v, and the complex masses m_i. It will turn out that the soliton structure is richest when we set the masses to be real and distinct:[3] $m_i = m_i^\dagger \neq m_j$ for $i \neq j$. In particular, this allows us to order the masses $m_i > m_{i+1}$.

The supersymmetric vacua of the theory exist when all terms in the potential can be set to zero. The presence of the vev v means that the q fields must be non-vanishing in

[1] All solutions described here can be uplifted to $d = 4+1$ dimensions where supersymmetry allows real masses for hypermultiplets.

[2] Our convention for the spacetime metric is $\eta_{MN} = \mathrm{diag}(-1,+1,+1,+1)$. We represent the fields in the vector multiplet in terms of matrices in the fundamental representation of $\mathrm{U}(N_c)$ whose generators T^a are normalized as $\mathrm{Tr}(T^aT^b) = \delta^{ab}$.

[3] This is entirely analogous to the situation with 't Hooft-Polyakov monopoles where the moduli spaces have largest dimension when only a single real adjoint scalar field has a vev.

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order that the second line of (2.1) vanishes. In turn, the presence of the masses m_i then requires that the ϕ field is non-zero so that the third line in (2.1) vanishes. The choice of a vacuum is equivalent to picking a set Σ of N_c distinct elements out of a possible N_f,

$$\Sigma = \{\sigma(a) : \sigma(a) \neq \sigma(b) \quad \text{for} \quad a \neq b\} \tag{2.2}$$

where $a = 1, \ldots, N_c$ runs over the colour index, while $\sigma(a) \in \{1, \ldots, N_f\}$. Up to a Weyl transformation, the vacuum is given by,

$$\phi = \mathrm{diag}(m_{\sigma(1)}, \ldots, m_{\sigma(N_c)}) \,, \qquad q^a{}_i = v\, \delta^a{}_{i=\sigma(a)} \tag{2.3}$$

supplemented by $\tilde{q} = 0$. For $N_f < N_c$ there are no supersymmetric vacua; for $N_f \geq N_c$, the number of vacua is $N_{\rm vac} = (N_f{-}\text{choose}{-}N_c) = N_f!/N_c!(N_f - N_c)!$. Each of these vacua is isolated and gapped. Parametrically, there are N_c^2 non-BPS massive gauge bosons and quarks with masses $m_\gamma^2 \sim e^2v^2 + |m_{\sigma(a)} - m_{\sigma(b)}|^2$ and $N_c(N_f - N_c)$ BPS massive quark fields with masses $m_q \sim |m_{\sigma(a)} - m_i|$ (with $i \notin \Sigma$).

The $\mathrm{U}(N_c)$ gauge symmetry and the $\mathrm{SU}(N_f)$ flavour symmetry of the theory are broken both explicitly by the masses m_i and spontaneously by the vev v^2. In each of the vacua, the pattern of symmetry breaking is[4]

$$\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \longrightarrow S\Big[\,\mathrm{U}(1)^{N_c}_{\rm diag} \times \mathrm{U}(1)^{N_f - N_c}\Big] \,. \tag{2.4}$$

Since the surviving unbroken group involves a simultaneous gauge and flavour rotation, the theory is said to lie in the colour-flavour-locked phase. However, note that in different vacua a different $\mathrm{U}(1)^{N_c} \subset \mathrm{U}(1)^{N_f - 1} \subset \mathrm{SU}(N_f)$ is locked with the gauge group, the choice determined by (2.2). This will play an important role when we discuss vortices connecting to domain walls.

For the solutions we consider, most of these fields will not play a role and we shall set them to zero at this stage. The reality of the masses allows us to restrict to a real adjoint scalar field: $\phi = \mathrm{Re}(\phi)$. We shall also set $\tilde{q}_i = 0$. One can show that neither $\mathrm{Im}(\phi)$ nor $\tilde{q}_i$ have zero modes on any of our soliton solutions. We shall further restrict to time independent solutions with vanishing electric fields $F_{0\alpha} = 0$ although there do exist dyonic versions of many of the solitons discussed below in which this constraint is relaxed. Truncating to the surviving fields, the bosonic[5] hamiltonian with which we shall work is,

$$\mathcal{H} = \frac{1}{2e^2}\,\mathrm{Tr}\left[B_\alpha^2 + (\mathcal{D}_\alpha\phi)^2\right] + \sum_{i=1}^{N_f} |\mathcal{D}_\alpha q_i|^2 + \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i + \frac{e^2}{2}\,\mathrm{Tr}\left[\left(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\right)^2\right] . \tag{2.5}$$

Here $B_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}F_{\beta\gamma}$ is the non-abelian magnetic field, with $\alpha = 1, 2, 3$ the spatial index. In the following sections we will embark on a tour of the wide variety of solutions on offer in this simple hamiltonian.

[4]The notation $S[\otimes_i \mathrm{U}(N_i)]$ means we project out the diagonal, central $\mathrm{U}(1)$ from $\otimes_i \mathrm{U}(N_i)$.

[5]When including fermionic fields, it is crucial to keep both the superpartners of q and $\tilde{q}$ to avoid anomalies. Moreover, both have zero modes on the soliton solutions described below.

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2.1 Massive gauged linear sigma-models

It is well known that the long-wavelength limit of certain supersymmetric gauge theories is described by a sigma-model on the Higgs branch of the theory which, in our case, is the cotangent bundle of the Grassmannian $T^\star G(N_c, N_f)$. This construction, often referred to as a gauged linear sigma model, has proven very useful in analysing non-linear sigma models, most notably in the context of two-dimensional gauge theories [1]. In the present case the masses for the scalar fields q and $\tilde{q}$ lift the Higgs branch. They can be thought of as inducing a potential on the sigma model target space of a specific form [2]. The long-wavelength limit of our theory is therefore given by a massive sigma model. Many (although not all) of the soliton solutions we will describe below have analogs in these massive sigma models which correctly capture the coarser features of the gauge theory configurations. Moreover, the solitons are usually simpler in the sigma-model context (for example, exact solutions can often be found in this limit) and, in most cases, were discovered there first. Since this connection between gauge theories and sigma models is important for our solitons, in appendix A we review how the sigma model emerges from the classical $e^2 \to \infty$ limit of the gauge theory.

3. Domain walls

The existence of multiple, gapped, isolated vacua is sufficient to guarantee the existence of co-dimension one domain walls. While domain walls in supersymmetric Wess-Zumino models have been studied in detail, the domain walls in the theory (2.1) have received the attention they deserve only recently. As we shall see, they have rather special properties. They were first studied by Abraham and Townsend in [3]. Subsequently, many people have examined their various properties in the sigma model limit [4]–[7], in abelian [8]–[13] and non-abelian [14, 15, 16] gauge theories, and in quantum theories [17, 18]. In this section we review the domain wall equations and present a new, simple method of classifying the domain walls in terms of the root lattice of the flavour group. As we shall see, this classification encodes much of the interesting information about the dynamics of domain walls.

Let us consider domain walls transverse to the x^3 direction. We pick a vacuum Σ at $x^3 = +\infty$ as determined by a set (2.2) and a distinct vacuum $\tilde{\Sigma}$ at $x^3 = -\infty$. The first order domain wall equations were first derived in [8] and can be determined by the usual Bogomoln'yi "completing-the-square" trick. If we restrict to configurations with $\partial_1 = \partial_2 = A_1 = A_2 = 0$, then the hamiltonian (2.5) can then be written as,

$$\begin{aligned}\int dx^3\,\mathcal{H} = &\int dx^3 \frac{1}{2e^2}\,\mathrm{Tr}\left[\left(\mathcal{D}_3\phi \mp e^2\left(\sum_i q_i q_i^\dagger - v^2\right)\right)^2\right] \pm \mathrm{Tr}\left[(\mathcal{D}_3\phi)\left(\sum_i q_i q_i^\dagger - v^2\right)\right] + \\ &+ \sum_i \left(|\mathcal{D}_3 q_i \mp (\phi - m_i) q_i|^2 \pm q_i^\dagger(\phi - m_i)\mathcal{D}_3 q_i \pm \mathcal{D}_3 q_i^\dagger(\phi - m_i) q_i\right) \\ \geq &\mp \left[\sum_i q_i^\dagger m_i q_i\right]_{-\infty}^{+\infty} \pm \left[\mathrm{Tr}\,\phi\left(\sum_i q_i q_i^\dagger - v^2\right)\right]_{-\infty}^{+\infty}\end{aligned}$$

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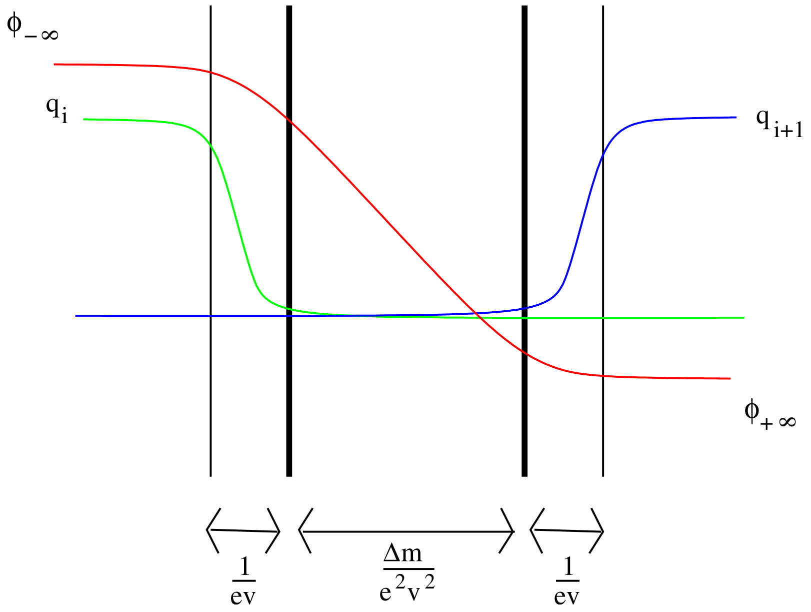

Figure 1: The profile of an elementary domain wall of type $\vec{g} = \vec{\alpha}_i$ when $e^2v^2 \ll |m_i - m_{i+1}|^2$. The fundamental fields q vary in the outer layer; the adjoint field ϕ varies in the inner layer. Notice that, with some of the q's vanishing, the gauge group is partially restored in the inner layer.

$$= \mp v^2 \, \mathrm{Tr}[\phi]_{-\infty}^{+\infty} = \mp v^2 \sum_{a=1}^{N_c} (m_{\sigma(a)} - m_{\tilde{\sigma}(a)}) \,. \tag{3.1}$$

For definiteness, let us choose our BPS domain walls to have $\Delta m < 0$ where $\Delta m = \sum_{i \in \Sigma} m_i - \sum_{i \in \tilde{\Sigma}} m_i$, so that we pick the upper signs in this equation. The inequality above is then saturated if the domain walls obey the first order Bogomoln'yi equations

$$\mathcal{D}_3 \phi = e^2 \left(\sum_i q_i q_i^\dagger - v^2 \right) \tag{3.2}$$

$$\mathcal{D}_3 q_i = (\phi - m_i) q_i \tag{3.3}$$

Configurations satisfying these equations are BPS, preserving 1/2 of the supersymmetries [4, 11]. While explicit solutions to these equations are not known in general, exact solutions can be found in the sigma-model limit $e^2 \to \infty$ [3, 9, 16] and, curiously, for specific finite values of e^2 [13]. Nevertheless, many properties of the solutions are well understood.

The structure of the domain wall profile at finite e^2 was studied in [19] and [11] and is rather interesting. It depends on the dimensionless parameter $e^2v^2/\Delta m^2$. When $e^2v^2 \gg \Delta m^2$, we are close to the sigma-model limit and we can refer to the exact solutions: one finds that all fields interpolate between their vacuum values over a width $1/\Delta m$. However, in the opposite limit $e^2v^2 \ll \Delta m^2$ things are somewhat different and the domain wall exhibits a three-layer structure. In the two outer layers, each of width $1/ev$, the fundamental fields drop to zero. In the inner layer which has width $\Delta m/e^2v^2$, the adjoint field ϕ interpolates between its two vacuum values. The configuration is shown in figure 1.

A heuristic understanding of how this structure arises can be gleaned as follows [11]: the components of the quark fields q_i and q_{i+1} which vary over the domain wall are non-BPS with masses scaling as ev. It is therefore to be expected that they vary with correlation length $1/ev$. But when the q_i vanish, equation (3.2) tells us that ϕ changes by Δm linearly with gradient e^2v^2, resulting in the larger inner segment of the wall profile.

3.1 Classification of domain walls

Domain walls in field theories are classified by a choice of vacuum $\tilde{\Sigma}$ and Σ at left and right infinity. However, in our theory the number of isolated vacua is $N_{\rm vac} = N_f!/N_c!(N_f - N_c)!$, which grows exponentially with N_f (fixing N_f/N_c), leading to a bewildering number of possible domain walls. To help alleviate the ensuing sense of confusion, we present here a simple classification of the possible walls which encodes many of their properties.

Firstly define the N_f-vector $\vec{m} = (m_1, \ldots, m_{N_f})$. From (3.1), the tension of a BPS domain wall is given by

$$T_{\rm wall} = -v^2\,{\rm Tr}[\phi]^{+\infty}_{-\infty} \equiv v^2\,\vec{m}\cdot\vec{g}\,>\,0 \tag{3.4}$$

where we have introduced the N_f-vector $\vec{g} \in \Lambda_R(su(N_f))$, the root lattice[6] of $su(N_f)$. Note that a choice of $\vec{g}$ does not imply a unique choice of vacua $\tilde{\Sigma}$ and Σ at left and right infinity. Nevertheless, domain walls which share the same $\vec{g}$ will share many of their essential properties. Note further that there do not exist domain walls for all $\vec{g} \in \Lambda_R(su(N_f))$: the only admissible vectors are of the form $\vec{g} = (p_1, \ldots, p_{N_f})$ with $p_i = 0$ or ± 1.

The first question that we wish to ask is whether there exist solutions to (3.2) and (3.3) for a given $\vec{g}$ and, if so, how many parameters does the solution have. In appendix B we perform an index theorem calculation to determine the number of zero modes to the domain wall equations. To express the answer, we firstly decompose $\vec{g}$ in terms of simple roots $\vec{\alpha}_i$,

$$\vec{g} = \sum_{i=1}^{N_f-1} n_i\vec{\alpha}_i$$

where $n_i \in \mathbf{Z}$. The basis of simple roots is determined by the vector $\vec{m}$ which defines the positive Weyl chamber when the flavour group is broken to the maximal torus: ${\rm SU}(N_F) \rightarrow {\rm U}(1)^{N_f-1}$. Hence, $\vec{m}\cdot\vec{\alpha}_i > 0$ for all i. Using our canonical basis of masses $m_1 > \ldots > m_{N_f}$, we have the simple roots $\vec{\alpha}_1 = (1, -1, 0, \ldots, 0)$ and $\vec{\alpha}_2 = (0, 1, -1, 0, \ldots, 0)$ all the way through to $\vec{\alpha}_{N_f-1} = (0, \ldots, 1, -1)$. The result of appendix B is that solutions to the BPS domain wall equations (3.2) and (3.3) exist only if $n_i > 0$ for all i. Furthermore, the number of zero modes of a solution is given by,

$$\#\text{ wall zero modes} = 2\sum_{i=1}^{N_f-1} n_i \tag{3.5}$$

This is the most general result on the counting of zero modes in this system. It is in agreement with earlier results including a Morse theory calculation in the sigma-model

[6] We ignore the factor of two distinction between the root and co-root lattice of the flavour group. This may become important in theories with non-simply laced flavour symmetries.

limit [5], an index theorem for the abelian theory [10] and an analysis of the zero modes in the non-abelian theory [16].

Equation (3.5) has a simple physical interpretation: a domain wall labelled by $\vec{g}$ can be thought of as constructed from $\sum_i n_i$ "elementary" domain walls, each labelled by a simple root α_i. Each of these elementary domain walls has two collective coordinates: a position in the x^3 direction, and a phase. The position is self-evident and ensures that there exist solutions which look like $\sum_i n_i$ well-separated elementary domain walls, with the distances between them arbitrary moduli. The phase is a little more subtle. It is required by supersymmetry (the domain wall preserves one half of $\mathcal{N}=2$ supersymmetry so must have a Kähler moduli space) and was first explained in [3]. Recall from (2.4) that each vacuum of our theory preserves a $G = S[\mathrm{U}(1)^{N_c}_{\mathrm{diag}} \times \mathrm{U}(1)^{N_f - N_c}]$ symmetry. However, the choice of which $\mathrm{U}(1)^{N_c} \subset \mathrm{SU}(N_f)$ gets locked with the gauge group differs from vacuum to vacuum. When a domain wall interpolates between two vacua, G is broken by the field configuration. Acting on the soliton with G results in Goldstone modes localised on the wall. There are also further massless, quasi-Goldstone modes predicted by the index theorem which do not arise from symmetries. More details on this phase collective coordinate can be found in [9, 11, 16].

The positions of the elementary domain walls are not allowed to be arbitrary: unlike solitons of higher co-dimensions, domain walls must obey at least some ordering on the line. In the abelian theory, this ordering is absolute and domain walls are not able to pass through each other [5, 9]. In the non-abelian theory, there is (literally) room for manoeuvre and certain domain walls are allowed to pass through each other [16]. Such walls are said to be "penetrable". Our simple classification of domain walls also captures this quality. Translating the analysis of [16], two elementary domain walls, $\vec{\alpha}_i$ and $\vec{\alpha}_j$, consecutive in space, may pass through each other whenever $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$. It is easy to motivate this result by noting that domain walls with this property sit in non-neighbouring parts of the flavour group and do not interact with each other; a linear superposition of two such domain walls yields a new solution.

Note that the expression (3.5) is reminiscent of the formula for the number of zero modes of monopoles in higher rank gauge groups [20]. Indeed, the relationship between domain walls of this type and magnetic monopoles has been commented on by several authors (see, for example, [3, 17]). The physical reason behind this similarity was explained in [21] and will be reviewed in section 4.

Finally a comment about the string theory realization of these domain walls. In [22] it was shown how the abelian domain walls appear as kinky D-strings in the D1-D5 system. More recently, the BPS wall solutions in the non-abelian gauge theory have been realized in a similar brane construction using fractional D-branes on orientifolds [23]. This brane construction is very convenient to understand and to visualize various properties of non-abelian walls such as penetrable versus impenetrable pairs of walls.

3.2 An example: $\mathrm{U}(2)$ with $N_f = 4$ flavours

The simplest non-abelian gauge theory whose domain wall system demonstrates many of the interesting features has $N_c = 2$ colours and $N_f = 4$ flavours. This was also the example discussed by Shifman and Yung in [14], although they considered the non-generic case of coincident masses so found somewhat different physics from that described below. The theory has $4!/2!2! = 6$ vacua, giving rise to $(6 \times 5)/2 = 15$ different domain walls. Let us describe the properties of all of these walls in terms of our classification.

For six choices of vacua at left and right infinity, there is only a single elementary domain wall interpolating between them. These choices are:

$$\begin{aligned}
\tilde{\Sigma} &= (1,2)\,, & \Sigma &= (1,3) & &\Rightarrow & \vec{g} &= \vec{\alpha}_2 \\
\tilde{\Sigma} &= (1,3)\,, & \Sigma &= (2,3) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 \\
\tilde{\Sigma} &= (1,3)\,, & \Sigma &= (1,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_3 \\
\tilde{\Sigma} &= (1,4)\,, & \Sigma &= (2,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 \\
\tilde{\Sigma} &= (2,3)\,, & \Sigma &= (2,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_3 \\
\tilde{\Sigma} &= (2,4)\,, & \Sigma &= (3,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_2\,.
\end{aligned}$$

Each of these domain wall systems has just two collective coordinates: a center of mass and a Goldstone phase.

For five choices of vacua at left and right infinity, the domain wall system decomposes into two elementary domain walls. These choices are:

$$\begin{aligned}
\tilde{\Sigma} &= (1,2)\,, & \Sigma &= (1,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_2 + \vec{\alpha}_3 \\
\tilde{\Sigma} &= (1,2)\,, & \Sigma &= (2,3) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 + \vec{\alpha}_2 \\
\tilde{\Sigma} &= (1,3)\,, & \Sigma &= (2,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 + \vec{\alpha}_3 \\
\tilde{\Sigma} &= (1,4)\,, & \Sigma &= (3,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 + \vec{\alpha}_2 \\
\tilde{\Sigma} &= (2,3)\,, & \Sigma &= (3,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_2 + \vec{\alpha}_3\,.
\end{aligned}$$

Each of these solutions has a regime where it looks like two well-separated domain walls. For four of the choices of vacua, the two domain walls cannot pass through each other: as they come together, they merge to form a single domain wall. The moduli space is smooth at this point [9]. However, the remaining choice of vacua is special: for $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_3$, we have $\vec{\alpha}_1 \cdot \vec{\alpha}_3 = 0$ and the domain walls are penetrable and may move through each other [16].

There is one further choice of vacua for which the domain wall system looks like two elementary domain walls. This is

$$\tilde{\Sigma} = (1,4)\,, \qquad \Sigma = (2,3) \qquad \Rightarrow \qquad \vec{g} = \vec{\alpha}_1 - \vec{\alpha}_3\,.$$

Note the minus sign! From the previous discussion, we learn that this system is to be thought of as a domain wall combined with an anti-domain wall. It breaks supersymmetry and there is no solution to the first order equations (3.2) and (3.3) with this choice of boundary conditions. This example of supersymmetry breaking was also noticed in [16].

There are two choices of vacua which have a system of three domain walls interpolating between them. They are

$$\begin{aligned}
\tilde{\Sigma} &= (1,2)\,, & \Sigma &= (2,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3 \\
\tilde{\Sigma} &= (1,3)\,, & \Sigma &= (3,4) & &\Rightarrow & \vec{g} &= \vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3\,.
\end{aligned}$$

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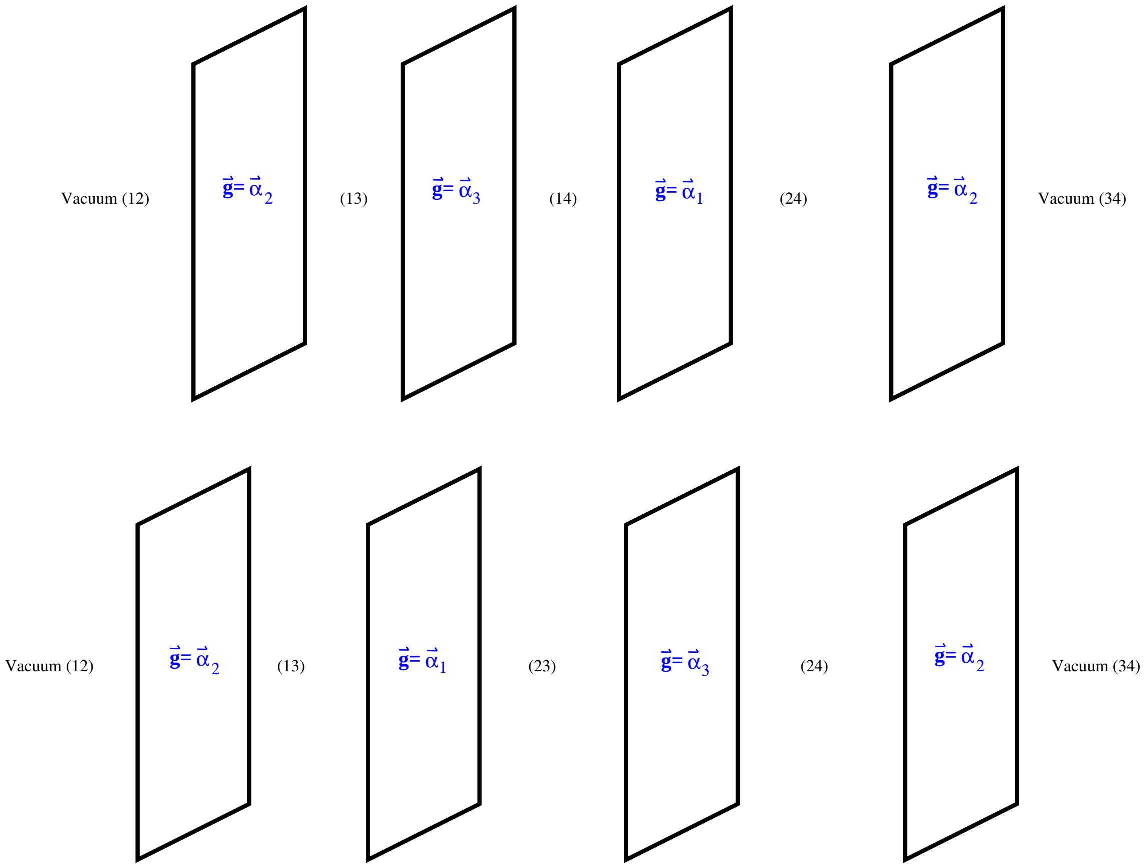

Figure 2: The two different orderings of the domain wall system $\vec{g} = \vec{\alpha}_1 + 2\vec{\alpha}_2 + \vec{\alpha}_3$. The two orderings are related by interchanging the inner two domain walls, an operation which is allowed since $\vec{\alpha}_1 \cdot \vec{\alpha}_3 = 0$. Notice that the vacuum in the middle of the system changes as the walls pass through each other.

In each of these systems, one domain wall is fixed in the ordering, while the other two may pass through each other. For the first set of vacua, moving in the direction of positive x^3, the ordering of domain walls is either $\vec{\alpha}_2 \to \vec{\alpha}_1 \to \vec{\alpha}_3$ or $\vec{\alpha}_2 \to \vec{\alpha}_3 \to \vec{\alpha}_1$. For the second system, the domain wall ordering can be either $\vec{\alpha}_1 \to \vec{\alpha}_3 \to \vec{\alpha}_2$ or $\vec{\alpha}_3 \to \vec{\alpha}_1 \to \vec{\alpha}_2$. Notice that this example reveals that the classification of domain wall systems in terms of $\vec{g}$ is *not* sufficient to determine the ordering of the walls: different systems with the same $\vec{g}$ may have different orderings.

Finally, the richest domain wall system occurs for a unique choice of vacua at left and right infinity:

$$\tilde{\Sigma} = (1,2)\,, \qquad \Sigma = (3,4) \qquad \Rightarrow \qquad \vec{g} = \vec{\alpha}_1 + 2\vec{\alpha}_2 + \vec{\alpha}_3$$

This system has 8 collective coordinates and may be decomposed into four elementary domain walls. The two $\vec{\alpha}_2$ domain walls are constrained to lie on the outside of the system, while the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls are free to exchange their positions in the middle. The two different orderings of the domain wall solutions are drawn in figure 2.

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3.3 The low-energy dynamics of walls

As we saw in equation (3.5), a domain wall system, classified by $\vec{g}$, has $\mathcal{I} = \sum_i n_i$ complex collective coordinates which we shall denote as X^m, $m = 1, \ldots, \mathcal{I}$. The X^m are coordinates on the domain wall moduli space $\mathcal{M}_{\rm wall}$. The low-energy dynamics of the domain walls is derived by promoting these collective coordinates to dynamical fields on the domain wall worldvolume, $X^m \to X^m(t, x^1, x^2)$, and plugging the resulting field content back into the lagrangian (2.1). Performing the integral over the x^3 direction transverse to the domain wall results in a $d = 2+1$ dimensional sigma-model with target space $\mathcal{M}_{\rm wall}$. The metric on $\mathcal{M}_{\rm wall}$ is determined by the kinetic terms of the original lagrangian [24] and, thanks to supersymmetry, is Kähler.

Consider a single elementary domain wall $\vec{g} = \vec{\alpha}_i$. The tension of such a wall is given by $T_{\rm wall} = v^2(m_i - m_{i+1})$ and the solution has a single complex collective coordinate:

$$X = x^3_{(0)} + i\theta \tag{3.6}$$

where $x^3_{(0)}$ is the center of mass of the domain wall in the x^3 direction, and θ is the phase of the wall arising from acting with the symmetry group (2.4) preserved in the vacuum. It has periodicity $\theta \equiv \theta + 2\pi v^2/T_{\rm wall}$. The low-energy dynamics of a single elementary wall is simply described by the following effective lagrangian on the wall worldvolume

$$-\mathcal{L}_{\rm wall} = \frac{1}{2} T_{\rm wall} \partial_\mu \bar{X} \partial^\mu X, \tag{3.7}$$

where $\mu = 0, 1, 2$ runs over the worldvolume of the domain wall. There is a dual formalism of the domain wall dynamics noticed in [25]. (See [11] for a careful discussion in the case with finite coupling e^2). In $d = 2+1$ dimensions, a periodic scalar field is dual to a U(1) photon:[7]

$$4\pi v^2 \partial_\mu \theta = \epsilon_{\mu\nu\rho} F^{\nu\rho} \tag{3.8}$$

where the normalisation is chosen so that a vortex in the θ field corresponds to a unit electric charge in the field strength F. We can therefore re-write the theory on the domain wall in terms of a single neutral scalar field $x^3_{(0)}$ and a U(1) gauge field $F_{\mu\nu}$

$$-\mathcal{L}_{\rm wall} = \frac{1}{2} T_{\rm wall} \partial_\mu x^3_{(0)} \partial^\mu x^3_{(0)} + \frac{(m_i - m_{i+1})}{16 v^2 \pi^2} F_{\mu\nu} F^{\mu\nu}\,. \tag{3.9}$$

As explained in [11], in the regime of parameters $e^2 v^2 \ll m^2$, the gauge field on the domain wall can also be understood in terms of the Dvali-Shifman localisation mechanism [26] since, as we see from figure 1, the profile of the domain wall results in an unbroken gauge group in the center. The fact that the gauge group is Higgsed in the bulk (as opposed to confined) means that the dual bulk gauge field is localised on the wall [27]. As a check of this interpretation, note that the effective $d = 2+1$ gauge coupling constant e^2_{2+1} in (3.9) scales as

$$\frac{1}{e^2_{2+1}} \sim \frac{(m_i - m_{i+1})}{v^2} \sim e^2 \times (\text{width of domain wall})\,.$$

[7] We use the same letter F to denote the U(N_c) bulk field and the U(1) worldvolume field and hope the context will suffice to alleviate any confusion.

The domain wall also has fermionic zero modes. These complete the above lagrangian to one with $\mathcal{N}=2$ supersymmetry in $d=2+1$ dimensions (that is, four supercharges). Thus the low-energy effective action for a single domain wall is a free $\mathcal{N}=2$ U(1) gauge theory. In section 6, we shall see how charged particles enter this picture.

For multiple domain walls, expressions for the metric on the moduli space $\mathcal{M}_{\rm wall}$ in abelian theories were computed in [9] (see also [18, 13]). For example, for two elementary domain walls $\vec{g}=\vec{\alpha}_i+\vec{\alpha}_{i+1}$, the moduli space is of the form $\mathcal{M}_{\rm wall}\cong \mathbf{R}\times(\mathbf{S}^1\times\tilde{\mathcal{M}})/\mathbf{Z_2}$ where $\tilde{\mathcal{M}}$ is a manifold with the topology of a cigar, corresponding to the relative position and phase of the two domain walls. For domain walls in non-abelian theories, a description of the domain wall moduli space was presented in [16] in terms of the underlying Grassmannian $G(N_c,N_f)$ with subspaces removed corresponding to "points at infinity".

4. Vortices

Our theory also admits another type of classical soliton solution: a vortex string. In the abelian case, these are simply the Abrikosov-Nielsen-Olesen vortices [28]. (For $N_f>1$ flavours, they are known as semi-local vortices [29]). In non-abelian gauge theories, the dynamics of vortices was studied in detail in [30, 31]. This section reviews the main results.

The Bogomoln'yi equations for vortices can be derived by completing the hamiltonian (2.5) in a different manner from the previous section. We will consider vortex strings lying in the x^3 direction, with the profile in the x^1,x^2 plane. So we restrict to configurations with $\partial_3=A_3=0$. The string carries magnetic flux $\int dx^1dx^2\,\mathrm{Tr}\,B_3=-2\pi k$ where $k\in\mathbf{Z}$. We have

$$\begin{aligned}\int dx^1dx^2\,\mathcal{H} &= \int dx^1dx^2\left[\frac{\mathrm{Tr}}{2e^2}\left(B_3\mp e^2\left(\sum_{i=1}^{N_f}q_iq_i^\dagger-v^2\right)\right)^2+\frac{1}{2e^2}\,\mathrm{Tr}[(\mathcal{D}_1\phi)^2+(\mathcal{D}_2\phi)^2]\right.\\ &\qquad\left.+\sum_{i=1}^{N_f}q_i^\dagger(\phi-m_i)^2q_i+\sum_{i=1}^{N_f}|(\mathcal{D}_1\mp i\mathcal{D}_2)q_i|^2\mp v^2\,\mathrm{Tr}\,B_3\right]\\ &\geq \mp v^2\int dx^1dx^2\,\mathrm{Tr}\,B_3=\pm 2\pi v^2k\,,\end{aligned}$$

where we choose the upper (lower) sign for $k>0$ ($k<0$). For the remainder of this section we restrict to $k>0$, but the possibility for both signs will become important when we consider the interaction of vortex strings with domain walls. The inequality above is saturated for BPS vortices of tension $T_{\rm vortex}=2\pi v^2|k|$ which satisfy the non-abelian first-order vortex equations,

$$B_3=e^2\left(\sum_{i=1}^{N_f}q_iq_i^\dagger-v^2\right)\,;\qquad \mathcal{D}_zq_i=0 \tag{4.1}$$

$$(\phi-m_i)q_i=0\,;\qquad \mathcal{D}_1\phi=\mathcal{D}_2\phi=0\,. \tag{4.2}$$

where $z=x^1+ix^2$ labels the plane transverse to the vortex string. The space of solutions to these equations is richest when the masses vanish: $m_i=0$. We will describe this situation first, then move on to explain how it is changed when the masses are turned on. When $m_i = 0$, the second line of equations (4.2) is trivially solved by $\phi = 0$ and we can focus on the first line. The number of zero modes of these equations was computed in [30], with the result

$$\#\text{ vortex zero modes} = 2kN_f \tag{4.3}$$

which generalises the answer in the abelian case computed in [32]. As with most BPS solitons, the proportionality to k can be understood as reflecting the fact that there exist solutions corresponding to k well-separated charge 1 vortices. Each of these has $2N_f$ zero modes which decompose as follows: two of these zero modes are simply the position of the vortex on the (x^1, x^2) plane. Of the remaining collective coordinates, some are Goldstone modes. To see this, note that when the masses vanish $m_i = 0$, the symmetry breaking pattern (2.4) is replaced by

$$\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \to S[\mathrm{U}(N_c)_{\mathrm{diag}} \times \mathrm{U}(N_f - N_c)]\,. \tag{4.4}$$

A single vortex generically breaks this surviving symmetry group, meaning that we can sweep out new vortex solutions by acting with it. Finally, there are also collective coordinates corresponding to approximate scaling modes (see for example [33]). We shall discuss the vortex moduli space in more detail presently.

How does this situation change when the masses $m_i \neq 0$? The answer to this question was given in [21] (see also [36, 37]). Given that some of the vortex zero modes are generated by acting with the symmetry (4.4) which is no longer available when masses are turned on (2.4), it should be immediately clear that much of the moduli space is lifted. In fact, the only surviving solutions to (4.1) and (4.2) are abelian vortex solutions embedded diagonally within the gauge group,

$$(B_3)^a{}_b = B^a_\star \delta^a{}_b\,, \qquad q^a{}_i = q^a_\star \delta^a{}_{i=\sigma(a)} \tag{4.5}$$

(no sum over a), where each pair $\{B^a_\star, q^a_\star\}$ for $a = 1, \ldots, N_c$ solves the abelian vortex equations. In this manner, a given vacuum contains N_c different types of vortex string, labelled by the diagonal element of the gauge group which carries the magnetic flux or, relatedly, by the flavour of quark field which has non-zero winding at infinity. The single topological quantum number k splits into N_c distinct quantum numbers $k_a \in \mathbf{Z} > 0$, where $\sum_a k_a = k$. (The gauge invariant object is $\int \mathrm{Tr}\,\phi B$). The vortices in each of these sectors have $2k_a$ zero modes, corresponding to the positions of k_a vortex strings in the (x^1, x^2) plane. All internal zero modes have been lifted by the masses. The N_c different strings are drawn for a given vacuum in figure 3. We shall refer to the string around which the phase of q_i winds as the "q_i-string". It only exists in a given vacuum if $i \in \Sigma$.

4.1 The low-energy dynamics of vortices

When the masses vanish ($m_i = 0$), the low-energy dynamics of vortex strings is derived by promoting the $2kN_f$ collective coordinates to dynamical fields on the string worldsheet (i.e. depending on t and x^3). Plugging the resulting configuration into the lagrangian

Figure 3: The N_c different types of vortices, labelled by the element of the magnetic field or, equivalently, by the flavour q_i which winds around the vortex. The N_c different vortex strings are mutually BPS.

(2.1) and performing the integral over the (x^1, x^2) plane results in a $d = 1+1$ dimensional sigma-model on the vortex moduli space $\mathcal{M}_{\rm vortex}$ where the result (4.3) tells us $\dim(\mathcal{M}_{\rm vortex}) = 2kN_f$. Since the vortex is 1/2-BPS, the $d = 1+1$ vortex theory has $\mathcal{N} = (2,2)$ supersymmetry (four supercharges); the superpartners of the bosonic zero modes are the (non-chiral) fermionic zero modes of the vortex string.

An example: for a single $k = 1$ vortex in U(N_c) gauge theory with $N_f = N_c$ flavours, the vortex moduli space is [30, 31]

$$\mathcal{M}_{\rm vortex} \cong \mathbf{C} \times \mathbf{CP}^{N_c-1} \tag{4.6}$$

where the Kähler class (or size) of $\mathbf{CP}^{N_c-1}$ is given by $r = 2\pi/e^2$. For higher k and higher N_f, the topology of the vortex moduli space was determined in [30].

When the masses are non-zero, we have seen that much of the vortex moduli space is lifted. This can be understood as introducing a potential on $\mathcal{M}_{\rm vortex}$ [21, 36, 37]. Here we present a description of the resulting dynamics on the vortex string worldsheet in terms of a $d = 1+1$ $\mathcal{N} = (2,2)$ gauged linear sigma model, derived using D-brane techniques in [30]. Consider a single $k = 1$ vortex string sitting in a given vacuum Σ. The worldvolume dynamics is described by a U(1) vector multiplet containing a gauge field G_{03} and a neutral complex scalar χ. Coupled to this are N_c chiral multiplets ψ_a of charge $+1$, and a further $(N_f - N_c)$ chiral multiplets $\hat{\psi}_m$ of charge -1. These charged chiral multiplets parameterise the internal degrees of freedom of the vortex. Finally, there is a single neutral chiral multiplet z containing the center of mass of the vortex. Denoting the lowest components of the chiral superfields by the same letters $\psi, \hat{\psi}$ and z, the bosonic part of the lagrangian describing the dynamics of a single vortex is:

$$-\mathcal{L}_{\rm vortex} = \frac{1}{2g^2}\left(G_{03}^2 + |\partial\chi|^2\right) + \sum_{a=1}^{N_c}\left(|\mathcal{D}\psi_a|^2 + |\chi - m_{\sigma(a)}|^2|\psi_a|^2\right) + |\partial z|^2 + \tag{4.7}$$

$$+\sum_{m=1}^{N_f-N_c}\left(|\mathcal{D}\hat{\psi}_m|^2+|\chi-m_{\hat{\sigma}(m)}|^2|\hat{\psi}_m|^2\right)+\frac{g^2}{2}\left(\sum_{a=1}^{N_c}|\psi_a|^2-\sum_{m=1}^{N_f-N_c}|\hat{\psi}_m|^2-r\right)^2,$$

where $\hat{\Sigma}$ is a set of $(N_f - N_c)$ elements $\hat{\sigma}(m)$, defined to be the complement of Σ in $\{1,\ldots,N_f\}$, and $|\partial\chi|^2 \equiv -(\partial_0\chi)^2 + (\partial_3\chi)^2$. This lagrangian contains three parameters: the masses $m_{\sigma(a)}$ and $m_{\hat{\sigma}(m)}$ are set equal to the masses m_i of the four-dimensional theory (strictly speaking, these are referred to as "twisted masses" in two dimensions [38]). The FI parameter r on the vortex worldsheet is given by [30]

$$r = \frac{2\pi}{e^2}\,. \tag{4.8}$$

Finally the gauge coupling is to be taken to infinity $g^2 \to \infty$, rendering the vector multiplet auxiliary since its kinetic term vanishes. As reviewed in appendix A, this ensures that attention is restricted to the Higgs branch of the theory given by the vanishing of the D-term ($\sum_a |\psi_a|^2 - \sum_m |\hat{\psi}_m|^2 = r$), modulo the U(1) gauge action. This Higgs branch is the vortex moduli space. The masses m_i induce a potential on this Higgs branch as anticipated in the above discussion.

An example: When $N_f = N_c$ we have no $\hat{\psi}$ fields. The D-term is simply $\sum_a |\psi_a|^2 = r$ which, after quotienting by the U(1) action $\psi_a \to \exp(i\alpha)\psi_a$ results in the Higgs branch $\mathbf{CP}^{N_c-1}$. If the masses are zero $m_i = 0$ then this Higgs branch is not lifted and, including the neutral field z, the vacuum moduli space of the theory (4.7) is given by $\mathcal{M}_{\rm Higgs} \cong \mathbf{C}\times\mathbf{CP}^{N_c-1}$ in agreement with the vortex moduli space (4.6).

When the twisted masses are turned on, the vortex theory (4.7) has N_c discrete vacua given by,

$$\text{Vacuum } a:\ \chi = m_{\sigma(a)}\,, \qquad |\psi_b|^2 = r\delta_{ab}\,, \qquad \hat{\psi}_m = 0\,. \tag{4.9}$$

This reflects the fact that, as explained above, the $k = 1$ vortex has N_c ground states determined by which scalar field q winds at infinity. The vacuum a of the vortex theory describes the situation in which the $q_{i=\sigma(a)}$ scalar field carries the winding and the magnetic field sits in the $a^{\rm th}$ diagonal component of B_3.

For k vortices, the low-energy dynamics is described by a U(k) gauge theory with an adjoint chiral multiplet, N_c fundamental chiral multiplets and $(N_f - N_c)$ anti-fundamental chiral multiplets. The derivation of this theory, together with a discussion of the physics it captures, can be found in [30].

5. Confined monopoles

So far we have discussed the domain wall and vortex string solutions of the theory. Now it is time to move onto monopoles. From the symmetry breaking pattern (2.4), there is no unbroken U(1) of the gauge symmetry and, correspondingly, no massless photon which can carry away the flux of a 't Hooft-Polyakov monopole. Nevertheless, as shown in [21], BPS monopoles do exist in this theory. Since they lie in the Higgs phase, their flux is carried away in collimated tubes. In other words, they are confined. In this section we review the results of [21]. The quantisation of these objects was considered in [36, 37] while discussions of confined monopoles in other theories can be found in [39].

The confined monopole is a 1/4-BPS object. The first order Bogomoln'yi equations can be once again derived from completing the square in the hamiltonian (2.5). This time we have two choices of $\pm$ signs. To make this clear, define ϵ and η to independently[8] take values ± 1. Then we can write

$$\begin{aligned}\int d^3x\,\mathcal{H} = & \int d^3x \frac{1}{2e^2}\,\mathrm{Tr}\Bigg[(\mathcal{D}_1\phi+\epsilon B_1)^2+(\mathcal{D}_2\phi+\epsilon B_2)^2 \\ & \qquad + \left(\mathcal{D}_3\phi+\epsilon B_3-\eta e^2\left(\sum_i q_i q_i^\dagger - v^2\right)\right)^2\Bigg] + \\ & + \sum_i |(\mathcal{D}_1-\epsilon\eta i\mathcal{D}_2)q_i|^2 + \sum_i |\mathcal{D}_3 q_i - \eta(\phi-m_i)q_i|^2 + \\ & + \mathrm{Tr}\left[-\eta v^2\partial_3\phi - \epsilon\eta v^2 B_3 - \frac{\epsilon}{e^2}\partial_\alpha(\phi B_\alpha)\right] \\ \geq & \int d^3x\,\mathrm{Tr}\left[-\eta v^2\partial_3\phi - \epsilon\eta v^2 B_3 - \frac{\epsilon}{e^2}\partial_\alpha(\phi B_\alpha)\right] \\ \equiv & \left[\int dx^1 dx^2\; T_{\rm wall}\right] + \left[\int dx^3\; T_{\rm vortex}\right] + M_{\rm mono}\,. \end{aligned} \tag{5.1}$$

In the final line the central charges for domain walls and vortices appear which are familiar from sections 3 and 4 respectively. The monopole central charge also makes an appearance,

$$M_{\rm mono} = -\frac{\epsilon}{e^2}\,\mathrm{Tr}\int \partial_\alpha(\phi B_\alpha)\,. \tag{5.2}$$

In this section, we shall set $T_{\rm wall}=0$ by working in the same vacuum Σ for all spatial infinity. In the following sections we shall relax this restriction to examine the most general configurations which involve monopoles, vortices and walls.

If we follow the convention of section 3 and 4, we set $\eta=\epsilon=+1$, have the Bogomoln'yi equations

$$B_1 = -\mathcal{D}_1\phi\,, \qquad B_2 = -\mathcal{D}_2\phi = 0\,, \qquad B_3 = -\mathcal{D}_3\phi + e^2\left(\sum_{i=1}^N q_i q_i^\dagger - v^2\right) \tag{5.3}$$

$$\mathcal{D}_1 q_i = i\mathcal{D}_2 q_i\,, \qquad \mathcal{D}_3 q_i = (\phi - m_i)q_i\,. \tag{5.4}$$

Other choices of minus sign are self-evident. The equations (5.3) and (5.4) are over-constrained: $2N_c(2N_c+N_f)$ real degrees of freedom, related by N_c^2 gauge transformations and $N_c(3N_c+4N_f)$ real differential equations. Consistency of the equations follows by noting the integrability condition [34]

$$[\mathcal{D}_1 - i\mathcal{D}_2, \mathcal{D}_3 - \phi + m_i] = -(B_1+\mathcal{D}_1\phi) + i(B_2+\mathcal{D}_2\phi) = 0\,. \tag{5.5}$$

[8]The sign η ($\epsilon\eta$) corresponds to the sign choice for the wall (the vortex) when Bogomoln'yi completion is performed.

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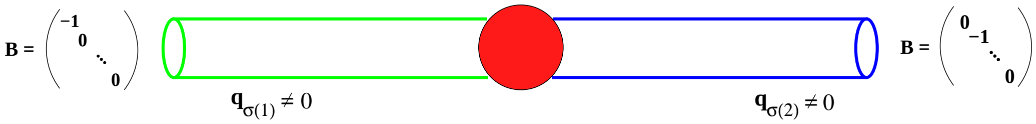

Figure 4: The confined monopole which is a source for magnetic flux $\int d^2x B = 2\pi\ \mathrm{diag}(1, -1, \ldots, 0)$.

This ensures that the two equations in (5.4) can be solved simultaneously for q_i. A similar integrability condition exists for 1/4-BPS equations describing instantons in the Higgs phase [35].

No explicit solutions to (5.3) and (5.4) are known. By examining the configuration from several perspectives, it was argued in [21] that solutions for equations (5.3) and (5.4) exist and several properties were derived. In [36] a radial ansatz was employed to find a solution perturbatively in $\Delta m^2/e^2v^2$. The solutions describe a magnetic monopole emitting two vortex flux tubes as sketched in figure 4. The energy of this entire configuration is infinite due to the presence of the two semi-infinite vortex strings. This is the statement that the monopole is confined. However, it still makes sense to talk about the finite mass $M_{\rm mono}$ of the monopole as an excitation over the energy of a single infinite, straight string. (This point will be made clearer shortly when we examine the monopole from the perspective of the vortex string). Recall that in the Coulomb phase the magnetic flux of the monopole escapes radially to infinity and is captured by the integral $M_{\rm mono} = -(1/e^2)\,\mathrm{Tr}\int d^3x\, \partial{\cdot}(\phi B)$ evaluated on the S^2_∞ boundary. In the present case, the magnetic flux does not make it to all points on the boundary, but is confined to two flux tubes which stretch in the $\pm x^3$ direction. Correspondingly, the integral should now be evaluated over two planes $\mathbf{R}^2_{\pm\infty}$ at $x^3 = \pm\infty$. Nevertheless, both integrals yield the same result.

The choice of minus signs ϵ and η when completing the square in (5.1) determine the direction in which the vortex strings are emitted from the monopole. For example, with $\epsilon = \eta = +1$ we have BPS vortices with negative flux $\int \mathrm{Tr}\, B_3 < 0$. The requirement that the monopole mass $M_{\rm mono} = -[\mathrm{Tr}(\phi \int B_3)]^{\infty}_{-\infty}$ is positive then fixes the direction in which the vortex strings lie. Since the masses are ordered as $m_i > m_{i+1}$, we learn that the confined monopole satisfying (5.3) and (5.4) must have a q_k vortex string emitted to the left and a q_j string emitted to the right with $k > j$. In terms of figure 4, we require $\sigma(1) > \sigma(2)$. Other choices of ϵ and η allow for a reversal of the string orientation but, crucially, a choice of η is required for compatibility for domain walls as we shall see in section 7.

Let us remind ourselves how to classify magnetic monopoles in higher rank gauge groups. The magnetic charge lives in the Cartan subalgebra $\mathrm{U}(1)^{N_c-1} \subset \mathrm{SU}(N_c)$ of the gauge group and is defined by a charge vector $\vec{h} \in \Lambda_R(su(N_c))$, the (co)root lattice[9] of the Lie algebra [40]. The vector $\vec{h}$ is N_c-dimensional, with integer components whose sum vanishes, imposing the requirement that the magnetic charge lives in $\mathrm{SU}(N_c) \subset \mathrm{U}(N_c)$. As is usual for BPS states, the mass of the monopole is determined by the topological charge

[9] Again, we ignore the factor of two difference between roots and co-roots which is unimportant for simply laced groups.

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$\vec{h}$. Recalling the vacuum expectation value $\phi = \text{diag}(m_{\sigma(1)}, \ldots, m_{\sigma(N_c)})$, we define the N_c dimensional vector $\vec{\phi} = (m_{\sigma(1)}, \ldots, m_{\sigma(N_c)})$. Then the mass of the monopole is

$$M_{\text{mono}} = \frac{2\pi}{e^2}\,\vec{h}\cdot\vec{\phi}\,. \tag{5.6}$$

As we mentioned above, this is the monopole mass in both the Coulomb and Higgs phases. Of course, in the latter phase this finite energy is supplemented by an infinite contribution from the vortex string.

Finally, we recall the result of E. Weinberg on counting the number of zero modes of the monopole [20]. If we decompose the magnetic charge into simple roots $\vec{\beta}_a$, with $a = 1, \ldots, N_c - 1$ such that $\vec{h} = \sum_a n_a \vec{\beta}_a$, then BPS monopoles exist for $n_a \in \mathbf{Z} > 0$. The number of zero modes of a solution *in the Coulomb phase* is $4\sum_a n_a$ [20]. As we commented in section 2, there are strong parallels between the classification and the counting of zero modes of monopoles and domain walls. This coincidence finds a compelling physical explanation when viewed from the perspective of the vortex string as we now review.

5.1 The view from the string

It is interesting to ask how the confined monopole looks from the worldsheet of the vortex string. In the previous section swe saw that the low-energy dynamics of a single string is described by a U(1) gauge theory with certain matter content. The lagrangian was given in equation (4.7). This theory has N_c vacuum states (4.9) corresponding to the N_c different vortex strings that exist in the theory (see figure 3). In summary, we have $d = 1 + 1$ dimensional theory on the string worldsheet with a set of isolated, discrete vacuum states. This immediately guarantees us that we have a new object: a kink on the string. This kink is the confined monopole [21].

In fact, the kink is of precisely the type discussed in section 2 ; the theory in question lives in $d = 1 + 1$ dimensions rather than $d = 3 + 1$, and has four supercharges rather than eight, but the Bogomoln'yi equations for the kink on the string have the same form as those describing the domain walls in section 2, namely

$$\partial_3\chi = -g^2\left(\sum_{a=1}^{N_c}|\psi_a|^2 - r\right), \qquad \mathcal{D}_3\psi_a = -(\chi - m_{\sigma(a)})\psi_a \tag{5.7}$$

supplemented with $\hat{\psi} = 0$. These are to be compared with (3.2) and (3.3). Note that the $\text{SU}(N_c)_{\text{diag}}$ group in spacetime has descended to the flavour group on the string worldsheet. The extra minus signs in (5.7) can be traced to the fact that the orientation of the emitted strings discussed above: the vortex theory is in the vacuum $\chi = m_k$ as $x_3 \to -\infty$ and $\chi = m_j$ as $x^3 \to +\infty$ with $k > j$ which implies that $m_k < m_j$.

From section 3, recall that we can classify kinks on the string in terms of a root vector of the flavour group which, in this case, is $\vec{h} \in \Lambda_R(su(N_c))$ where $[\chi]^{+\infty}_{-\infty} = \vec{\phi}\cdot\vec{h}$. The mass of the kink is then given by (c.f. equation (3.4))

$$M_{\text{kink}} = r\,[\chi]^{+\infty}_{-\infty} = r\,\vec{\phi}\cdot\vec{h} = \frac{2\pi}{e^2}\,\vec{h}\cdot\vec{\phi}$$

where, in the last equation, we have used the expression for r given in equation (4.8). Comparing with the mass of the monopole calculated in four dimensions (5.6), we find the important equation,

$$M_{\rm kink} = M_{\rm mono}\,.$$

Although we will not describe the details here, it was shown in [36, 37], following the calculations of [17], that this equation continues to hold when all quantum effects are taken into account: note that the left-hand-side is computed in a $d = 1+1$ dimensional theory, while the right-hand-side is computed in a $d = 3+1$ dimensional theory. The vortex gives a quantitative correspondence between the theories in different dimensions.

It is simple to see that the kink carries the quantum numbers of the monopole. It interpolates between two vacua (4.9) on the vortex worldsheet which, in $d = 3+1$ dimensional spacetime, means it joins two vortex strings with their magnetic field embedded in different components of the gauge group (4.5). The kink can be thought of as absorbing the magnetic field from the left, and splitting out a different magnetic field to the right. In other words, it is a source for the magnetic field with charge $\vec{h}$: it is a magnetic monopole.

For multiple k strings, the U(k) gauge theory describing their dynamics was described in [30] and section 4. When $k < N_c$, there exist vacua of the worldsheet theory in which the adjoint chiral multiplet has vanishing vev, and the strings are localised on top of each other in space. At this point, the kinks on the string are identical to those discussed in section 3 and their zero modes are given by the computation on the appendix B, summarised in (3.5). As we already noted, the classification and zero mode structure of domain walls is entirely analogous to that of magnetic monopoles. Now the physical reason behind this similarity is clear and is the slogan of this section: the kinks on the vortex string *are* magnetic monopoles.

Finally, one could also ask about the euclidean vortex on the euclidean vortex string. This corresponds to a Yang-Mills instanton trapped inside the vortex string [37]. An explicit realization of this idea was recently obtained and the relation between the monopole in the Higgs phase and the instanton in the Higgs phase was clarified [35].

6. D-branes

As we have seen, our theory contains both domain walls and vortex strings. In fact, it was noticed some years ago that the domain walls are D-branes for the vortex strings: the vortices may terminate on the wall where their end is electrically charged under the U(1) gauge field on the domain wall worldvolume. This was first shown in [25] for abelian theories in the limit $e^2 \to \infty$ where explicit solutions were found. The generalisation to finite gauge coupling was explored in detail by Shifman and Yung in [11, 14] and the solutions were examined in various limits. Solutions in non-abelian theories in the $e^2 \to \infty$ limit were discussed in [15]. We note also that D-branes in the field theoretic context have been shown to exist in other semi-classical models [41] and also arise as a strong coupling phenomenon in which QCD flux-tubes can end on domain walls [42].

In this section, we will explore several properties of the D-brane system and point out the existence of a new contribution to the BPS energy density which was overlooked in

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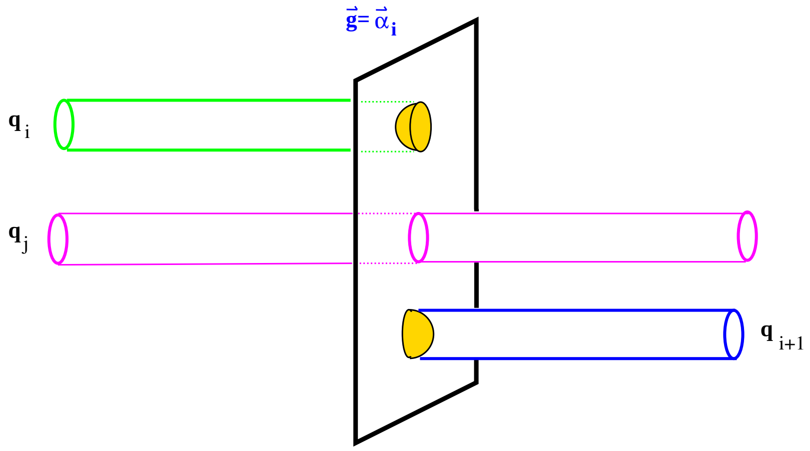

Figure 5: For the $\vec{\alpha}_i$ elementary domain wall, the q_i string may end on the left and the q_{i+1} string may end on the right. All q_j strings, for $j \neq i, i+1$ exist in both left and right vacua and pass right through the domain wall. The nodes represent the finite binding energy.

previous studies: a negative binding energy where the string attaches to the domain wall. It is unusual to encounter negative contributions to central charges, but we shall argue that it is perfectly sensible in this setting.

6.1 D-branes

It turns out that the equations describing a vortex string ending on a wall are identical to those describing confined monopoles (5.3) and (5.4). In fact, these equations were first discovered by Shifman and Yung in context of abelian field theoretic D-branes [11]. The different solutions follow from imposing different boundary conditions: for D-branes we require that both $[\mathrm{Tr}\,\phi]^{+\infty}_{-\infty}$ and $\int \mathrm{Tr}\, B$ are non-vanishing.

Recall from section 3 that we have $N_f!/N_c!(N_f - N_c)!$ vacua, providing a large selection of domain walls. Recall also from section 4 that in each vacua we have N_c different types of vortices. Thus the first question we ask is simple:[10] Which type of vortex can end on which type of domain wall? Since we have shown that any system of domain walls can be decomposed into a number of elementary domain walls labelled by a simple root of $\Lambda_R(su(N_f))$, let us restrict attention to such an elementary object, labelled by

$$\vec{g} = \vec{\alpha}_i = (0, \ldots, 1, -1, \ldots, 0)$$

where the entry "1" sits in the i^{th} position. This means that the vacuum $\tilde{\Sigma}$ to the left of the domain wall differs from the vacuum Σ to the right by a single element: $i \in \tilde{\Sigma}$ but $i \notin \Sigma$, while $(i+1) \in \Sigma$ but $(i+1) \notin \tilde{\Sigma}$. Now we can state our result: from the vacuum on the left, only the vortex associated to q_i can end on the domain wall; from the vacuum on the right only the vortex associated to q_{i+1} is allowed to end. We depict this in figure 5. In fact, it is simple to see this result and requires only a quick inspection of the relevant

[10] For the case of U(2) gauge group with four flavours, this question was answered in [14] in a non-generic case with degenerate masses. Our answers agree in the relevant limit.

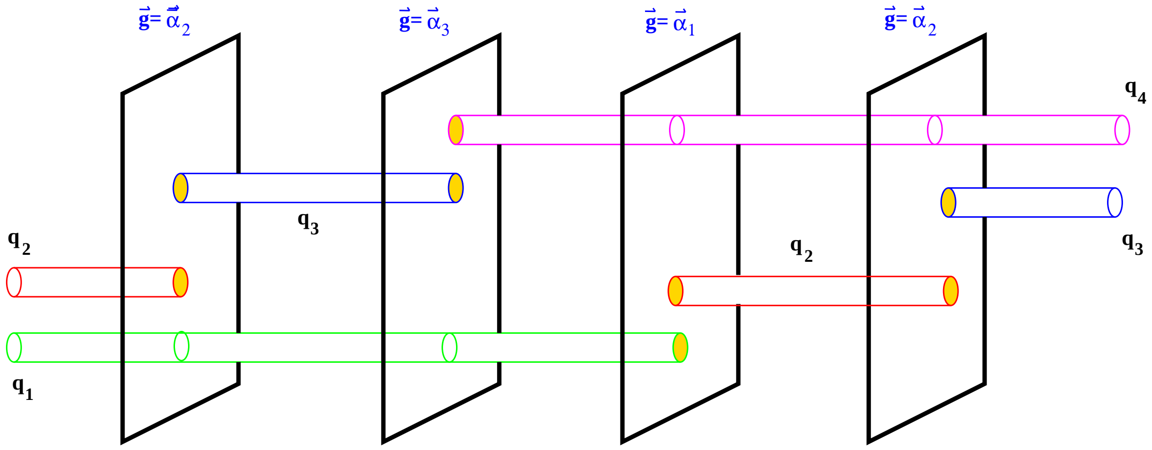

Figure 6: Which string on which wall: the allowed string ends for the $\vec{g} = \vec{\alpha}_1 + 2\vec{\alpha}_2 + \vec{\alpha}_3$ domain wall system in the U(2) theory with $N_f = 4$ flavours.

Bogomoln'yi equations from (5.3) and (5.4)

$$\mathcal{D}_3 q_i = (\phi - m_i) q_i \,.$$

Since we are dealing with an elementary domain wall, only a single diagonal component of ϕ (let's call it ϕ_a) picks up a profile such that $\phi_a \to m_{\tilde{\sigma}(a)} = m_i$ as $x^3 \to -\infty$, while $\phi_a \to m_{\sigma(a)} = m_{i+1}$ as $x^3 \to +\infty$. But, in the vacuum $\tilde{\Sigma}$, all vortices other than that built around q_i have their winding in the different part of the gauge group and so it cannot change as we move in the x^3 direction and cross the domain wall.

The rule for which strings can end on which domain walls is invariant under permutations of the domain wall ordering (when allowed). For example, in figure 6 we show which strings can end on the maximal domain wall system of the U(2) gauge theory with $N_f = 4$. As the inner two domain walls pass through each other, the strings ending on them are unaffected.

Notice in particular that the rules described above allow a vortex string to be suspended between adjacent elementary walls *only if* the walls cannot pass each other. For example, in figure 6 we see that it is not possible to suspend a string between the two inner walls $\vec{\alpha}_1$ and $\vec{\alpha}_3$. This means we do not have to ask the question about what happens to these finite segments of string as the domain walls change position. However, in section 7 we will discuss a slight variant of this set-up where this situation does arise.

Finally, let us mention that the simple picture of a rigid string ending on a rigid domain wall as shown in the figures is not really accurate. In fact, the string pulls the domain wall and, if the number of strings ending on the left and right is not equal, the domain wall bends asymptotically in a logarithmic fashion [25, 11]. This is familiar behaviour for co-dimension two objects (for example: the fundamental string ending on a D2-brane, or the D4-brane ending on an NS5-brane in IIA string theory at finite coupling). This logarithmic bending will play a crucial role when we come to study the binding energy.

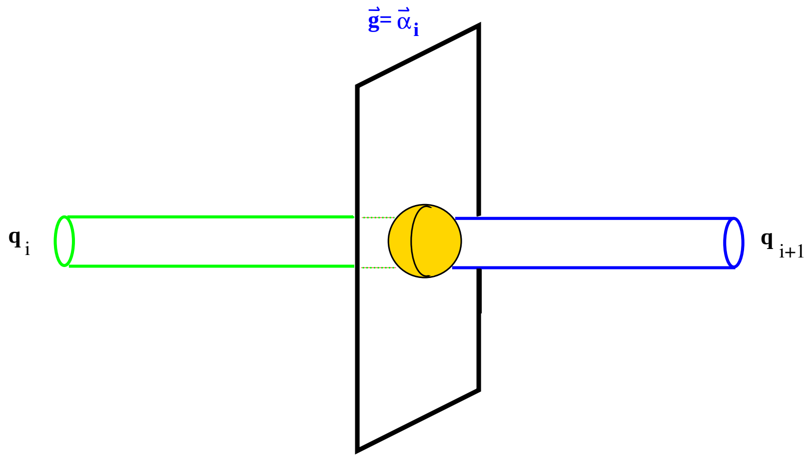

Figure 7: When colinear strings end, a binding energy equivalent to a negative mass monopole appears in the middle.

6.2 Binding energy

Now let us turn to the binding energy, a negative contribution to the total energy of the system. Since this binding energy occurs for BPS solitons, it can be easily seen by examining the central charge or, equivalently, the topological charge left after completing the square in the hamiltonian. We saw from equation (5.1) that we are left with three such terms: the wall charge, the vortex charge and the monopole charge. The binding energy is associated with the latter; more specifically, and rather curiously, it is one half of a negative mass BPS monopole. The field configuration connecting the string to the wall is known in the condensed matter literature as a boojum.

The sign of the binding energy is fixed by the requirement that the tensions of the domain walls and vortex strings are positive. From the completion of the square in (5.1), we see that this requires us to pick signs for η and $\epsilon\eta$ respectively; the sign of the monopole central charge cannot then be altered and is fixed to be ϵ as in (5.2). We will now show that the monopole contributes a negative energy. We first fix our choice of BPS domain walls and BPS vortex strings by choosing $\epsilon = \eta = +1$ in (5.1). Start by considering a wall with two vortex strings ending on it; one from the left and one from the right. Let's start with the strings colinear as illustrated in figure 7. Calculating the contribution to the energy from the monopole central charge, we have

$$E_{\text{binding}} = -\frac{1}{e^2}\int dx^1 dx^2 \left[\mathrm{Tr}(\phi B_3)\right]_{-\infty}^{+\infty} = -\frac{2\pi}{e^2}(m_i - m_{i+1}) < 0 \tag{6.1}$$

where the contribution comes from the different values adopted by ϕ to the left and right of the domain wall, and is non-vanishing despite the fact that B_3 remains constant at left and right infinity. Notice that even though this is a finite contribution to the energy, we can cleanly separate it from the infinite contribution of the vortex strings and the (infinite)^2 contribution of the domain wall since it is the only object whose energy depends parametrically on the gauge coupling e^2. Usually when we encounter a negative central charge, it means we have computed the contribution for an anti-soliton instead of a soliton;

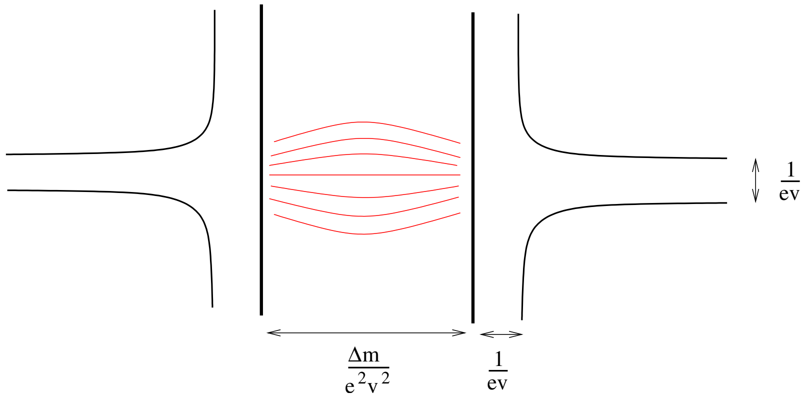

Figure 8: Colinear vortex strings ending on the domain wall. In the centre of the domain wall, the magnetic flux is free to spread out.

we should change some minus signs around and find the correct, positive mass for the BPS state. However, this is not the case here. The minus signs are fixed by requiring that the domain wall and vortex contributions are positive: $T_{\text{wall}} > 0$ and $T_{\text{vortex}} > 0$. The negative contribution to the binding energy (6.1) then follows. The same result can be easily seen to hold for other choices of $\epsilon, \eta = \pm 1$.

How should we interpret the binding energy (6.1)? We can find a heuristic explanation by looking at the profile of the domain wall in the limit $ev \ll \Delta m$ where E_{binding} becomes large. This profile was drawn in figure 1. Now consider what it looks like with two strings attached. The two outer layers of the domain wall are the same width as the vortex string and within both objects one of the Higgs fields q drops exponentially to zero. This shared property makes it natural for the vortex string to smoothly merge into the outer layer of the domain wall. But what occurs in the middle layer? The magnetic flux $\text{Tr} \int dx^1 dx^2\, B_3$ must be conserved in the middle of the domain wall. But here the flux is free to spread out. The BPS soliton must therefore look like figure 8. The binding energy can be understood as the energy difference between the spreading flux in the middle layer, shown in figure 8, and the energy contained in a collimated vortex tube of the same length. The latter has already been accounted for in the topological charge $-v^2\, \text{Tr} \int B_3$, but this is overcounting; the binding energy remedies this. To see that the parametric scaling agrees, note that the vortex tube stretched across the inner layer would have energy

$$E_{\text{vortex}} \sim T_{\text{vortex}} \frac{\Delta m}{e^2 v^2} \sim \frac{\Delta m}{e^2} \tag{6.2}$$

while the actual energy from the flux can be readily estimated. If, for some value of x^3, the flux spreads over an area A then the magnetic field scales as $B_3 \sim 1/A$. The effective tension (energy density per unit x^3 length) of the flux tube in that region is

$$\mathcal{E}_{\text{flux}} \sim \frac{1}{e^2}\, \text{Tr} \int dx^1 dx^2\, B_3^2 \sim \frac{1}{e^2 A}\,. \tag{6.3}$$

When the flux is confined in the vortex, it spreads over an region $A \sim 1/e^2 v^2$, resulting in the vortex tension $T_{\text{vortex}} \sim v^2$. Inside the domain wall, flux is unconfined and it spreads over a greater region $A \sim \alpha/e^2v^2$ for some $\alpha > 1$. We therefore have the expectation

$$E_{\text{binding}} = (\mathcal{E}_{\text{flux}} - T_{\text{vortex}})A \sim \left(\frac{1}{\alpha} - 1\right)\frac{\Delta m}{e^2v^2} \sim -\frac{\Delta m}{e^2} < 0 \tag{6.4}$$

which is indeed confirmed by the central charge (6.1).

The phenomenon of a negative BPS binding energy has been encountered previously. Wess-Zumino models in $d = 3 + 1$ admit BPS spoke-like solutions consisting of multiple domain walls meeting at a junction. The supersymmetry algebra includes a central charge for the $d = 1+1$ intersection of the domain walls [47]. An exact solution for the junction of three walls was constructed and the new central charge turned out to give a negative contribution to the energy [48]. As with our situation, the central charge has the interpretation of a BPS binding energy.

So much for colinear vortices. What is the binding energy for a single string ending on a domain wall? The answer, as we shall now argue, is that a single string has binding energy

$$E_{\text{binding}} = -\frac{1}{2}\,\frac{2\pi}{e^2}\,(m_i - m_{i+1}) = -\frac{1}{2}M_{\text{mono}}\,. \tag{6.5}$$

At first glance, this seems rather unlikely! How can this fractional (negative) mass arise from the topological charge given the well-known quantisation conditions? To see how this feat is achieved, consider a situation where the vortex string ends from the left. When evaluated at $x^3 \to -\infty$, the topological charge gives

$$\frac{1}{e^2}\int dx^1dx^2\,\mathrm{Tr}(\phi B_3)\Big|_{-\infty} = -\frac{2\pi m_i}{e^2}\,. \tag{6.6}$$

At right infinity it naively appears that there is no contribution since there are no vortex strings there. However, this is not correct because, as we mentioned previously, the domain wall bends logarithmically away from the terminating string. Denoting the center of mass of the domain wall as $x^3_{(0)}$, the asymptotic profile of the D-brane is $x^3_{(0)} \sim \log|z|$ [25, 11]. Crucially for us, this means that the magnetic flux is deposited on the domain wall by the vortex string, and then carried to $x^3 \to +\infty$ by this logarithmic bending, resulting in a contribution to the topological charge. Since the domain wall interpolates from $\phi_a = m_i$ at $x^3 = -\infty$ to $\phi_a = m_{i+1}$ at $x^3 = +\infty$, in the center of the domain wall the scalar field takes the average value $\phi_a = \frac{1}{2}(m_i + m_{i+1})$. Thus the contribution to the charge at right infinity is

$$-\frac{1}{e^2}\int dx^1dx^2\,\mathrm{Tr}(\phi B_3)\Big|_{+\infty} = +\frac{2\pi}{e^2}\,\frac{1}{2}\,(m_i + m_{i+1})\,. \tag{6.7}$$

Adding together the two contributions (6.6) and (6.7), we find that the biding energy of the string ending on the wall is indeed given by half the mass of the monopole.

Let us comment that the presence of the binding energy differentiates these field theoretic D-branes from those of type-II string theory. Naively it appears that stretched strings become massless, and then tachyonic, as domain walls approach and overlap. Were they to truly become unstable, the result would be to Higgs the axial U(1) on the worldvolume under which they are charged, thereby reducing the number of collective coordinates [27].

Since the index theorem of appendix B shows no sign of this behaviour, it seems that the domain walls cannot approach close enough to allow the string to become tachyonic. (Recall that penetrable domain walls, which can pass through each other, cannot have strings stretched between them).

More interesting is the possibility that the stretched strings become exactly massless, resulting in non-abelian symmetry enhancement. This was conjectured to occur in [14] in the non-generic case of degenerate masses, although the non-abelian nature was exhibited only at the linearised level. It would be interesting to examine the domain wall moduli space in the non-abelian theory to find evidence for non-abelian symmetry enhancement.

6.3 An aside: boojums in superfluid ^{3}He

The term boojum was introduced to physics by Mermin in the context of superfluid ^{3}He [43]. If ^{3}He-A is placed in a closed, spherical container, the boundary conditions imposed at the surface require that the order parameter becomes singular at a point. This point — the boojum — emits a vortex texture into the bulk of the fluid. More pertinent to the present discussion, boojums also occur on the AB-interface [44]. This interface is a domain wall separating the two phases of superfluid ^{3}He. A global vortex in the ^{3}He-B phase may end on the domain wall, where it terminates on a boojum. The binding energy of our system can be thought of as a negative mass boojum. In ^{3}He, as in our case, the boojum is a close relative of the hedgehog (or monopole). For a nice introduction to this subject, see the entertaining book [45]. Boojums are also observed in other condensed matter systems, including cholesteric and smectic liquid crystals [46].

6.4 The view from the domain wall

In section 3, we reviewed the low-energy dynamics on an elementary domain wall. From this perspective, the vortex strings are rather simple to see: they correspond to spike-like solutions to the low-energy effective action [25, 11], analogous to the "BIon" solutions of the Born-Infeld action for D-branes in string theory [49]. For k_R strings ending from the right at position z_p^R (recall $z = x^1 + ix^2$) and k_L strings ending from the left at positions z_q^L, the solution on the domain wall worldvolume is

$$\frac{2T_{\rm wall}}{v^2}\, X(z) = \log\left(\frac{\prod_{q=1}^{k_L}(z - z_q^L)}{\prod_{p=1}^{k_R}(z - z_p^R)}\right) \tag{6.8}$$

where $X = x^3_{(0)} + i\theta$ was defined in equation (3.6). Note that the domain wall bends logarithmically as $|z| \to \infty$ unless $k_L = k_R$. The holomorphic property of this solution is sufficient to ensure that it preserves 1/2 of the four supercharges on the domain wall worldvolume, corresponding to a 1/4-BPS state from the $d = 3+1$ dimensional perspective.

Consider a single vortex string ending from the left. As described in detail in [25] and [11], as we travel once around infinity $z \to e^{2\pi i}z$, the phase angle on the domain wall worldvolume shifts: $\sigma \to \sigma + 2\pi$. In other words, the vortex string is a global vortex for σ. Switching to the dual field strength F on the worldvolume of the domain wall (3.8), we see that the vortex string is electrically charged.

The low-energy action on the domain wall given in (3.9) is valid only at suitably large length scales. For example, it certainly does not hold at length scales comparable to the width of the domain wall which, as we reviewed in section 1, varies from $1/\Delta m$ to $\Delta m/e^2v^2 \gg 1/\Delta m$ as we dial the dimensionless parameter $ev/\Delta m$ from large to small. This limitation of the domain wall dynamics can be seen in the solution (6.8) which describes an increasingly narrow spike as $z \to z_q^L$ or $z \to z_p^R$ whereas, in reality, the vortex string stabilises at a width $1/ev$. This breakdown of the domain wall dynamics at short distances also manifests itself in another way. It is a simple matter to evaluate the energy of the solution (6.8) using the domain wall theory (3.7). One finds $k_L + k_R$ ultraviolet divergences which can be interpreted as the infinite vortex strings. For $k_L \neq k_R$, there is also an infra-red divergence arising from the asymptotic bending of the domain wall. However, there is no hint of the binding energy. Indeed, we have seen that this binding energy occurs over a distance comparable to width of the domain wall and so might suspect that the worldvolume theory is too coarse to capture it. However, given the BPS nature of the binding energy there may exist a formulation in which it is seen by the domain wall dynamics. We leave this as an open question.

7. Rules of interaction

In this final section we would like to discuss further rules of interaction between the various solitons described above. We will now consider all possible combinations of objects — walls, strings, and monopoles — and ask how they join together and under what circumstances different objects may pass through each other. Our analysis is based purely on the energetics of the BPS solitons and we find a unique set of rules *under the assumption* that there exist solutions to the Bogomoln'yi equations (5.3) and (5.4) corresponding to far-separated solitons of various types. While such a "no-force" condition is a familiar consequence of the BPS property of solitons, counterexamples are known (see for example [5]) and it is important to stress that while our crude analysis results in a consistent and unique picture of the interactions, it falls short of proving that the rules we describe occur dynamically. Further work attempting to demonstrate the existence of the solutions is in progress.

Firstly consider solitonic configurations which involve both domain walls and confined monopoles. The domain walls are 1/2-BPS; the confined monopoles are 1/4-BPS. When combined, it turns out that the configuration still preserves 1/4 of the supersymmetry [15], rather than 1/8 as one might expect. This can be seen when completing the square in (5.1) by noting that one has only two choices of $\pm$ signs, ϵ and η rather than three. Fixing η and $\epsilon\eta$ determine the half of the supercharges preserved by the domain wall and vortex respectively. The two may coexist peacefully, preserving 1/4 of the supersymmetry. We have no further $\pm$ signs to play with and only monopole for which $M_{\rm mono} > 0$ preserve this remaining supersymmetry.

In terms of gamma matrices, this discussion is somewhat simpler [34]. The domain wall and vortex preserve 1/2 of the supercharges, determined by the positive eigenspace of a suitably defined gamma matrix: $\xi = \gamma_{\rm wall}\xi$ and $\xi = \gamma_{\rm vortex}\xi$. The fact that these solitons can be mutually 1/4 BPS is the statement that $[\gamma_{\rm wall}, \gamma_{\rm vortex}] = 0$. Finally, the monopole

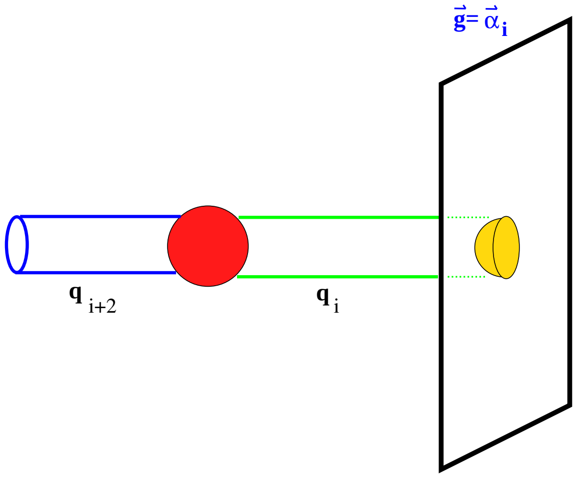

Figure 9: A configuration allowed by supersymmetry. Is the position of the monopole a modulus, or is it attracted or repelled by the wall?

breaks no further supersymmetries because its gamma matrix is related to the other two: $\gamma_{\rm mono} = \gamma_{\rm wall}\gamma_{\rm vortex}$.

The fact that 1/4 of the supersymmetry is preserved has an important consequence: in the presence of a domain wall, a confined monopole may only be oriented one way in the x^3 direction and still preserves supersymmetry. This follows immediately from the η and ϵ assignments in (5.1). If we choose BPS domain walls with $\eta = +1$ and BPS vortex strings with $\epsilon = +1$ (corresponding to $\mathrm{Tr}\int B_3 < 0$) then we already saw in section 5 that the Bogomoln'yi equation (5.3) and (5.4) require that a q_k string should be emitted to the left and the q_j string to the right of the wall, if $k > j$. To see that this is the orientation compatible with domain walls, simply note that the Bogomoln'yi equations for the confined monopole (5.3) and (5.4) contain within them the BPS domain wall equations (3.2) and (3.3). While the condition above was derived for BPS vortices defined as $\int \mathrm{Tr}\, B_3 < 0$, it can be easily checked that the same holds for BPS anti-vortices with $\int \mathrm{Tr}\, B_3 > 0$.

Now we understand which orientation of confined monopoles is compatible with the orientation of vortices, we can try to build a soliton configuration consisting of walls, vortices and monopoles. These putative BPS configurations should satisfy the Bogomoln'yi equations (5.4) with the appropriate topological charges arising from the boundary conditions. For example, if the left vacuum $\tilde{\Sigma}$ supports vortex strings of the q_{i+2} and q_i type, a possible BPS configuration would be a q_{i+2} string entering from the left, transforming into a q_i string by the presence of a monopole, and subsequently ending on a $\vec{g} = \vec{\alpha}_i$ domain wall. If this monopole were in the Coulomb phase it would be able to decompose into two fundamental monopoles but, in the vacuum $\tilde{\Sigma}$, this is not possible since there is no q_{i+1} vortex string to take away the intermediate component magnetic flux. (By definition of the $\vec{g} = \vec{\alpha}_i$ domain wall, only the right vacuum supports such a string). The configuration is shown in figure 9. We stress that we do not know whether the distance between the monopole and domain wall is a modulus of such a solution as suggested by the BPS bound and a naive application of the "no-force" folk theorem. We would be interested in knowing if this is indeed the case.

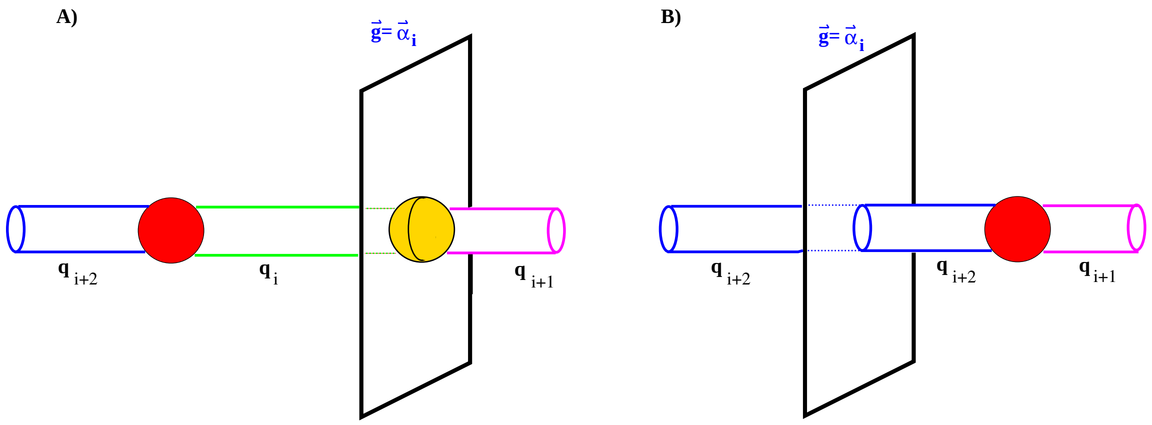

Figure 10: A monopole passes through the domain wall, soaking up the negative binding energy in the process.

Let us now ask whether monopoles are permitted to pass through domain walls. Obviously they can only pass through if there is a colinear string waiting on the other side onto which they can slide. Suppose that we have a $\vec{g} = \vec{\alpha}_i$ domain wall in which both left and right vacua support q_{i+2} vortex strings. We can reconsider the situation of figure 9, but now where there is a q_{i+1} string attached to the right as shown in figure 10a. Energetically, it is now possible for the monopole to pass through the domain wall. As it does so, it absorbs the negative binding energy, leaving behind a q_{i+2} string which penetrates the domain wall unopposed. In the right vacuum Σ, sits a monopole of lower charge, connecting the q_{i+2} string to the q_{i+1} string. It's mass is lower than the original monopole by an amount equal to the binding energy. This final configuration is shown in figure 10b.

Another question: what becomes of monopoles that are trapped between domain walls as the walls approach and pass through each other? The answer depends on whether there are further strings extending to the left and right providing an escape route for the monopole. Let's first examine the case where there can be no escape. Consider two penetrable elementary domain walls $\vec{\alpha}_i$ and $\vec{\alpha}_j$ with $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$. Assume that $i > j$. (In fact $i > j+1$ since they are penetrable). We saw in section 6 that a simple vortex string cannot be suspended between adjacent, penetrable walls. But the BPS conditions allow us to stretch a string between the walls with a monopole suspended in the middle. This is shown in figure 11. When the $\vec{\alpha}_i$ domain wall is to the left of the $\vec{\alpha}_j$ domain wall, as in figure 11a, we see that the q_{i+1} string is to the left of the q_j string. By our criterion above, this ensures that the objects in the configuration are mutually BPS for vortex strings satisfying $\int \mathrm{Tr}\, B_3 < 0$. The monopole charge for this configuration has three contributions: from the left and right binding energies, and from the central monopole.

$$\begin{aligned}\frac{e^2}{2\pi} M_{\rm mono} &= -\frac{1}{2\pi}\int dx^1 dx^2\, \mathrm{Tr}(\phi B_3)\Big|_{-\infty}^{+\infty} \\ &= -\frac{1}{2}(m_i - m_{i+1}) + (m_j - m_{i+1}) - \frac{1}{2}(m_j - m_{j+1}) > 0\,. \end{aligned} \tag{7.1}$$

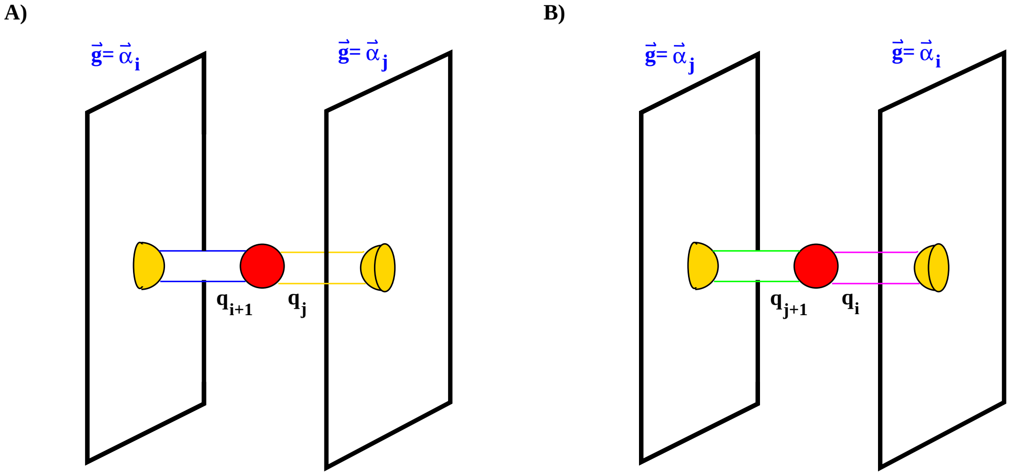

Figure 11: Crushing the monopole. The two configurations with the domain walls interchanged carry the same asymptotic charge, but only the configuration on the left preserves supersymmetry.

Now consider attempting to exchange the position of the domain walls. What becomes of the monopole and string? Since the charge (7.1) is measured at infinity, it cannot simply disappear. Indeed, there is a configuration carrying the correct charge, drawn in figure 11*b*. However, to get the minus signs right, we require that the strings in this configuration are anti-vortices, with $\mathrm{Tr} \int B_3 > 0$. The BPS monopole charge for figure 11*b* is given by,

$$\frac{e^2}{2\pi} M_{\rm mono} = \frac{1}{2}(m_j - m_{j+1}) + (m_{j+1} - m_i) + \frac{1}{2}(m_i - m_{i+1}) \tag{7.2}$$

which agrees with (7.1). However, from the discussion above we see that this configuration breaks all supersymmetry. The orientation of the confined monopole is not compatible with the presence of domain walls. We therefore conclude that the transition from figure 11*a* to figure 11*b* is not smooth. It seems likely that no solution of the form 11B exists.

It is an interesting phenomenon that a pair of penetrable walls can no longer exchange positions once a vortex with a monopole stretches between them. This is made more plausible when we recall that the vortex string causes the two domain walls to bend logarithmically to infinity. In figure 11*a*, the $\vec{\alpha}_i$ domain wall bends to left infinity, while in figure 11*b* it would bend to right infinity. Thus, despite appearances, the two configurations are actually separated by an infinite distance in field space. Both this, and the supersymmetry argument above, point to the existence of a minimal separation between the two domain walls in figure 11*a* when a confined monopole is hung between them.

Finally, let's turn to an example where the monopole can escape the crush by returning to the case of four walls in the U(2) gauge theory with $N_f = 4$ flavours. An interesting monopole configuration is drawn in figure 12*a*. We want to know what happens as the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls change position. A configuration with the same monopole charge and the walls interchanged is drawn in figure 12*b*. This time, and in contrast to the situation in figure 11, both configurations preserve the same set of supercharges, suggesting that

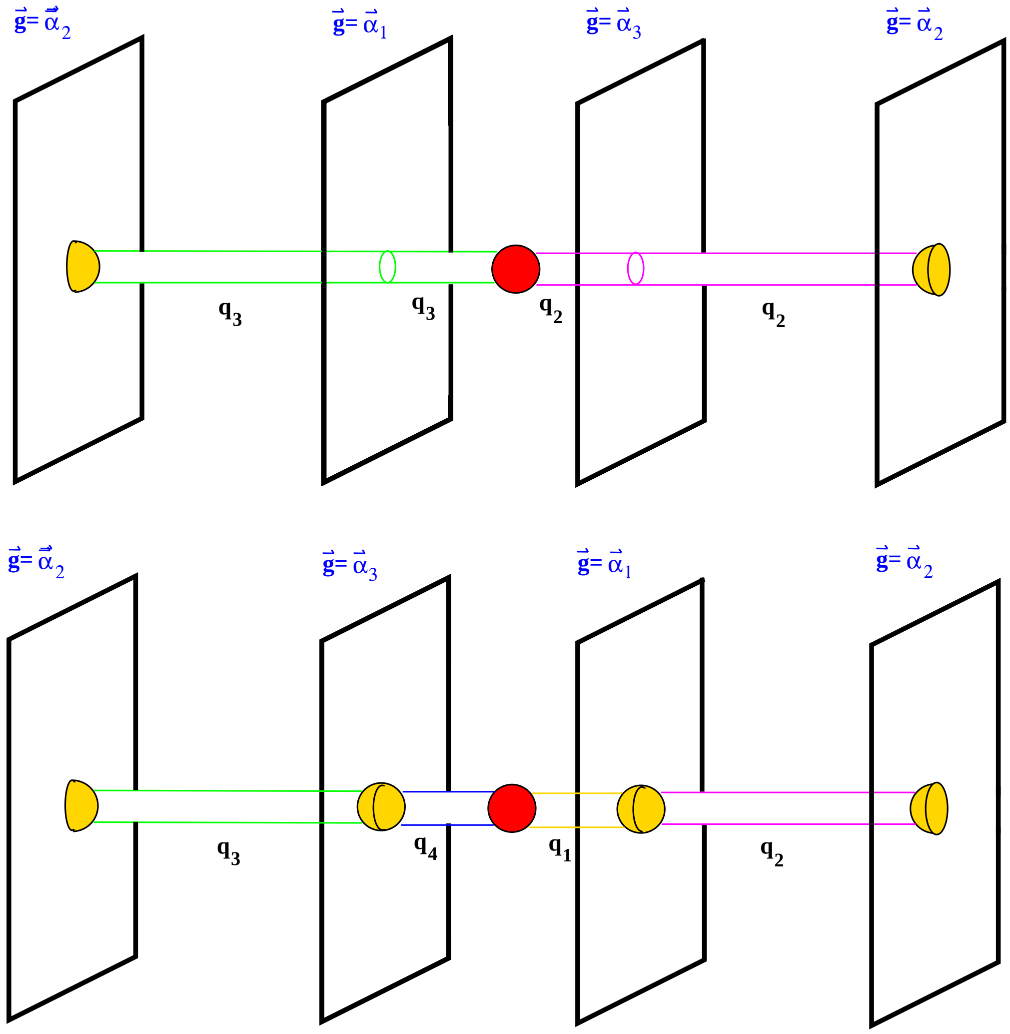

Figure 12: A monopole trapped between two domain walls as they pass. In this case the monopole can escape. The final, supersymmetric, configuration is depicted in the lower figure after the inner two domain walls have exchanged positions.

the transition is smooth. Indeed, it is also possible to get from figure 12a to 12b by first moving the monopole out of harms way; the monopole may slide either left or right, past the $\vec{\alpha}_1$ or the $\vec{\alpha}_3$ wall, promoting itself to a higher charge monopole while leaving behind a negative binding energy. We can now allow the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls to pass through each other, and finally slide the monopole back into the middle vacuum. The end result is that the monopole has grown to a higher charge object, with its excess energy accounted for in two binding energies on the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls.

A. Gauge theories and massive sigma models

As described in [1], the long-wavelength limit of the gauge theory (2.5) is a massive sigma-model with target space given by the Higgs branch of the gauge theory. Many of the solitons described below are somewhat simpler in the sigma-model and, indeed, were first discovered in that context. Here we briefly recall how the relationship to the sigma model works.

The relevant limit is at strong coupling $e^2 \to \infty$. Here the D-terms written in the second line of (2.1) are imposed as constraints. Ignoring the mass terms for now, the moduli space of solutions to this equations is given by the possible choices of N_c orthonormal vectors in $\mathbf{C}^{N_f}$. The $\mathrm{U}(N_c)$ gauge transformations act by rotating this basis of vectors meaning that the space of gauge invariant solutions is given by the set of N_c-dimensional planes in $\mathbf{C}^{N_f}$. This is simply the symplectic quotient construction of the Grassmannian $G(N_c, N_f)$. Thus the Higgs branch is

$$\mathcal{M}_{\mathbf{Higgs}} \cong G(N_c, N_f)\,. \tag{A.1}$$

(Including the $\tilde{q}$ fields in the analysis, and working with the D-term of (2.1) results in the cotangent bundle $\mathcal{M}_{\text{Higgs}} = T^\star G(N_c, N_f)$. However, all soliton solutions live on the zero-section $G(N_c, N_f)$ which is the sigma-model translation of the fact that we can ignore the $\tilde{q}$ fields. In a slight abuse of notation, we will therefore continue to refer to the Grassmannian as the Higgs branch).

In the absence of the masses, we would simply be left with a sigma model with target space $\mathcal{M}_{\text{Higgs}}$. The metric on this space descends from the kinetic terms of the gauge theory via the Kähler quotient construction. We denote this metric as g_{pq} where $p, q = 1, \ldots, 2N_c(N_f - N_c)$. In the presence of the masses, the sigma-model kinetic term is augmented with a potential given by

$$V = \sum_{i=1}^{N_f} q_i^\dagger (\phi - m_i)^2 q_i$$

where ϕ is to be taken to solve its equation of motion which, in the strong coupling limit, is algebraic $\sum_i (\phi - m_i) q_i q_i^\dagger = 0$. In fact, there is a very nice geometrical interpretation for the potential V when written in terms of the sigma-model. To see this, it is necessary to discuss the symmetries of the Higgs branch. When the masses are set to zero, the $\mathrm{SU}(N_f)$ flavour symmetry has a natural action on $\mathcal{M}_{\mathbf{Higgs}}$ resulting in an $\mathrm{SU}(N_f)$ isometry of the target space. The masses — and hence the potential V – break this to $\mathrm{U}(1)^{N_f-1}$. Consider the associated $N_f - 1$ mutually commuting holomorphic Killing vectors on $\mathcal{M}_{\text{Higgs}}$. In fact, it will prove useful to overcount by embedding $\mathrm{SU}(N_f) \subset \mathrm{U}(N_f)$ in the natural fashion, resulting in N_f Killing vectors K_i on $\mathcal{M}_{\text{Higgs}}$ satisfying the constraint $\sum_i K_i = 0$. Then it is not hard to show that the potential V can be rewritten as

$$V = \sum_{i,j=1}^{N_f} (m_i K_i^p)\,(m_j K_j^q)\, g_{pq}$$

Potentials of this form were first described by Alvarez-Gaumé and Freedman who showed that this is the unique potential allowed by sigma-models with 8 supercharges [2] (see also [4] for a very slight generalisation).

We see that in the language of the sigma-model, the vacua of the theory $V = 0$ are the fixed points of the $\mathrm{U}(1)^{N_f-1}$ action on $G(N_c, N_f)$. On general grounds (for example, using Morse theory) it is known that there are $N_{\rm vac} = \chi(G(N_c, N_f)) = N_f!/N_c!(N_f - N_c)!$ such fixed points, in agreement with the gauge theory analysis above.

B. The index theorem for domain walls

In this appendix we perform the calculation of the Callias index theorem in order to count the zero modes of a domain wall solution interpolating between different vacua. For domain walls in abelian gauge theories, the index theorem was computed by K. Lee in [10] while for domain walls in the sigma-model limit a Morse theory argument was given in [5]. Here we give the calculation for the non-abelian gauge theory and derive the result (3.5).

We let q denote a $N_c \times N_f$ matrix. Similarly, let m denote the $N_f \times N_f$ matrix which is simply $m = \mathrm{diag}(m_1, \ldots, m_{N_f})$. Then the domain wall Bogomoln'yi equations can be written as

$$\mathcal{D}_3\phi = e^2(qq^\dagger - v^2) \tag{B.1}$$

$$\mathcal{D}_3 q = \phi q - qm\,. \tag{B.2}$$

We linearise around a background solution and write $\delta\phi = \hat{\phi}$ with similar notation for other fields. We have

$$\mathcal{D}_3\hat{\phi} - i[\hat{A}_3, \phi] = e^2(\hat{q}q^\dagger + q\hat{q}^\dagger) \tag{B.3}$$

$$\mathcal{D}_3\hat{q} - i\hat{A}_3 q = \phi\hat{q} + \hat{\phi}q - \hat{q}m \tag{B.4}$$

which are to be supplemented by a gauge fixing condition which naturally comes from Gauss' law and is taken to be

$$\mathcal{D}_3\hat{A}_3 - i[\phi, \hat{\phi}] = ie^2(q\hat{q}^\dagger - \hat{q}q^\dagger) \tag{B.5}$$

We can combine (B.3) and (B.5) by introducing the complex adjoint field $\xi = \phi + iA_3$ in terms of which the equations become,

$$\partial_3\hat{\xi} + [\xi^\dagger, \hat{\xi}] - 2e^2\hat{q}q^\dagger = 0 \tag{B.6}$$

$$\partial_3\hat{q} - \xi\hat{q} + \hat{q}m - \hat{\xi}q = 0\,. \tag{B.7}$$

We wish to determine the number of solutions to the two complex linearised Bogomoln'yi equations (B.6) and (B.7). We can rescale the fields to absorb $2e^2$ in eq. (B.6) without affecting the number of solutions. In the following we take $2e^2 \to 1$. To make the notation of matrix multiplication somewhat simpler, it is useful to take the transpose of these equations. We define $h = q^T$, an $N_f \times N_c$ matrix, and $\zeta = \xi^T$, an $N_c \times N_c$ matrix. The advantage of working with the transpose matrices is that our two equations may now be combined into a single matrix equation,

$$\Delta \begin{pmatrix} \hat{\zeta} \\ \hat{h} \end{pmatrix} = 0 \tag{B.8}$$

where Δ is an $(N_f+N_c)\times(N_f+N_c)$ square matrix operator given by

$$\Delta=\begin{pmatrix}\partial_3-\zeta_a^\dagger & -h^\dagger\\ -h & \partial_3-\zeta_R+m\end{pmatrix}. \tag{B.9}$$

The operator $\zeta_a^\dagger$ acts by adjoint action, while the operator ζ_R acts by right multiplication (as indeed it must by simply examining the indices). The operator Δ acts on a vector space of $(N_c+N_f)\times N_c$ matrices $\psi=(\alpha,\beta)^T$ where the inner product between $\psi=(\alpha,\beta)^T$ and $\phi=(\gamma,\delta)^T$ is defined by

$$\begin{aligned}(\phi,\psi)&\equiv \mathrm{Tr}\left[(\gamma,\delta)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right]\equiv(\gamma,\alpha)+(\delta,\beta)\\ &\equiv\int dx^3\left(\mathrm{Tr}\left[\gamma^\dagger(x^3)\alpha(x^3)\right]+\mathrm{Tr}\left[\delta^\dagger(x^3)\beta(x^3)\right]\right).\end{aligned} \tag{B.10}$$

Therefore the adjoint operator $\Delta^\dagger$ is given by

$$\Delta^\dagger=\begin{pmatrix}-\partial_3-\zeta_a & -h^\dagger\\ -h & -\partial_3-\zeta_R^\dagger+m\end{pmatrix}. \tag{B.11}$$

To proceed, we firstly show that the adjoint operator $\Delta^\dagger$ has no zero modes. Then the norm of $\Delta^\dagger$ acting on an arbitrary $(N_c+N_f)\times N_c$ matrix $(\alpha,\beta)^T$ is given by

$$\begin{aligned}|\Delta^\dagger\begin{pmatrix}\alpha\\ \beta\end{pmatrix}|^2&=|\partial_3\alpha-[\zeta^\dagger,\alpha]-h^\dagger\beta|^2+|\partial_3\beta-\beta\zeta+m\beta-h\alpha|^2\\ &=|\partial_3\alpha-[\zeta^\dagger,\alpha]|^2+|h^\dagger\beta|^2+|\partial_3\beta-\beta\zeta+m\beta|^2+|h\alpha|^2+\text{X-terms}\end{aligned} \tag{B.12}$$

where the cross terms are given by,

$$\begin{aligned}\text{X-terms}&=(\partial_3\alpha+[\zeta,\alpha],h^\dagger\beta)+(h^\dagger\beta,\partial_3\alpha+[\zeta,\alpha])+\\ &\quad+(\partial_3\beta+\beta\zeta^\dagger-m\beta,h\alpha)+(h\alpha,\partial_3\beta+\beta\zeta^\dagger-m\beta)\\ &=(\partial_3\alpha,h^\dagger\beta))+(\alpha,[\zeta^\dagger,h^\dagger\beta])+(\beta,h(\partial_3\alpha+[\zeta,\alpha]))+\\ &\quad+(\partial_3\beta,h\alpha)+(\beta,h\alpha\zeta-mh\alpha)+(\alpha,h^\dagger(\partial_3\beta+\beta\zeta^\dagger-m\beta))\\ &=(\alpha,(-\partial_3h^\dagger+\zeta^\dagger h^\dagger-h^\dagger m)\beta)+(\beta,(-\partial_3h+h\zeta-mh)\alpha)\end{aligned}$$

To reach the second and third equalities, we have made use of the cyclic property of the trace and integrated by parts, respectively. The equations in the parentheses are simply the transpose of the domain wall equations (B.2) and therefore vanish on the background soliton. Since these cross-terms vanish, the action of $\Delta^\dagger$ can be written as a sum of total squares as in (B.12), which includes the term $|h\alpha|^2$ and, for h of maximal rank ensures that any zero mode of $\Delta^\dagger$ must have $\alpha=0$. To show that the zero mode also requires $\beta=0$ is only slightly more involved. The term $|h^\dagger\beta|^2$ can vanish for $\beta=-iTh$, where T is a hermitian $N_f\times N_f$ matrix. With this ansatz, one can show that the matrix T must solve something akin to a lax equation: $\partial_3T=[m,T]$ which, imposing the boundary condition $T=0$ at $x^3=\pm\infty$ requires $T=0$. We conclude $\Delta^\dagger$ has no zero modes.

The number of complex zero modes of Δ is defined by $\mathcal{I} = \lim_{M^2\to 0}\mathcal{I}(M^2)$ where the regulated index $\mathcal{I}(M^2)$ is

$$\mathcal{I}(M^2) = \mathrm{Tr}\left(\frac{M^2}{\Delta^\dagger\Delta + M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\Delta\Delta^\dagger + M^2}\right). \tag{B.13}$$

Clearly $\mathcal{I}$ counts the number of zero modes of Δ minus the number of zero modes of $\Delta^\dagger$. But since, as we have seen, the latter operator has vanishing kernel, $\mathcal{I}$ counts the number of parameters of the domain wall equations as required. Using standard Callias index theorem techniques (see for example [20] for application to monopoles, and [10] for application to domain walls), we can rewrite the index as

$$\mathcal{I}(M^2) = -\frac{1}{2}\left[J(x,M^2)\right]_{x=-\infty}^{x=+\infty}$$

where the current J is defined as

$$J(x,M^2) = -\,\mathrm{tr}\langle x|\Delta\frac{1}{\Delta^\dagger\Delta + M^2} + \Delta^\dagger\frac{1}{\Delta\Delta^\dagger + M^2}|x\rangle\,. \tag{B.14}$$

Here tr denotes the trace over colour and flavour indices, as well as over the 2×2 entries of Δ.

We first need to compute $\Delta\Delta^\dagger$ and $\Delta^\dagger\Delta$. In general these are somewhat complicated matrices, but our task is made a little simpler by the fact that we only need to evaluate them in a vacuum state. This means that we may set $A = 0$ or, in our current notation, $\zeta = \zeta^\dagger = \phi^T$. We can also insist that derivatives of the background field vanish: $\partial\zeta = \partial h = 0$. Finally, from the F-term condition, we also have that $h\zeta = mh$. With these constraints, the two matrix products may be easily evaluated to give

$$\Delta^\dagger\Delta\Big|_{\mathrm{vacuo}} = \Delta\Delta^\dagger\Big|_{\mathrm{vacuo}} = \begin{pmatrix} -\partial_3^2 + \zeta_a^2 + h^\dagger h & 0 \\ 0 & -\partial_3^2 + (\zeta_R - m)^2 + hh^\dagger \end{pmatrix}. \tag{B.15}$$

Let's firstly examine the current with the expressions we derived above, neglecting the trace over flavour indices and the integral over space. We have

$$\begin{aligned}
&\mathrm{tr}_2\,\mathrm{Tr}_{N_c}\left[\Delta\frac{1}{\Delta^\dagger\Delta + M^2} + \Delta^\dagger\frac{1}{\Delta\Delta^\dagger + M^2}\right] = \\
&\quad = \mathrm{Tr}_{N_c}\left[(\partial_3 - \zeta_a^\dagger)\frac{1}{\Theta_1 + M^2} + (\partial_3 - \zeta_R + m)\frac{1}{\Theta_2 + M^2} - \right.\\
&\qquad\quad \left. - (\partial_3 + \zeta_a)\frac{1}{\Theta_1 + M^2} - (\partial_3 + \zeta_R^\dagger - m)\frac{1}{\Theta_2 + M^2}\right]
\end{aligned}$$

where, for simplicity of notation, we have defined the operators Θ_1 and Θ_2 arising in the expression (B.15) for $\Delta^\dagger\Delta$ and $\Delta\Delta^\dagger$

$$\Delta^\dagger\Delta\Big|_{\mathrm{vacuo}} = \Delta\Delta^\dagger\Big|_{\mathrm{vacuo}} = \begin{pmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{pmatrix}. \tag{B.16}$$

We see that the ∂ derivatives in the numerator explicitly cancel, while the adjoint action ζ_a vanishes upon taking the trace over the gauge group. Recalling that $\zeta = \zeta^\dagger$ in vacuo, we're left with,

$$\mathrm{tr}_2\,\mathrm{Tr}_{N_c}\left[\Delta\frac{1}{\Delta^\dagger\Delta + M^2} + \Delta^\dagger\frac{1}{\Delta\Delta^\dagger + M^2}\right] = -2\,\mathrm{Tr}_{N_c}\left[(\zeta_R - m)\frac{1}{\Theta_2 + M^2}\right]. \tag{B.17}$$

Now we can evaluate the differential operator ∂_3^2 in momentum space to obtain

$$J(x, M^2) = \mathrm{Tr}_{N_c}\,\mathrm{Tr}_{N_f}\left[\frac{(\zeta_R - m)}{\sqrt{(\zeta_R - m)^2 + hh^\dagger + M^2}}\right] \tag{B.18}$$

which, in turn, yields an expression for the number of zero modes:

$$\mathcal{I} = \mathcal{I}_\infty - \mathcal{I}_{-\infty} = -\frac{1}{2}\,\mathrm{Tr}_{N_c}\,\mathrm{Tr}_{N_f}\left[\frac{(\zeta_R - m)}{\sqrt{(\zeta_R - m)^2 + hh^\dagger}}\right]_{-\infty}^{+\infty} \tag{B.19}$$

where $\mathcal{I}_{\pm\infty}$ denotes the expression in the bracket evaluated at $x^3 \to \pm\infty$. We are nearly there. All that is left is some algebraic manipulation. Recall from (2.3) that as $x^3 \to +\infty$, we have $\phi = \mathrm{diag}(m_{\sigma(1)}, \ldots, m_{\sigma(N_c)})$ and $q^a{}_i = v\,\delta^a{}_{\sigma(a)}$. The combination of q that appears in the expression for the index is

$$hh^\dagger = q^T(q^T)^\dagger = v^2 I_\infty \tag{B.20}$$

where the $N_f \times N_f$ matrix I_∞ has only non-zero diagonal entries which are given by $(I_\infty)_{ii} = 1$ if $i \in \Sigma$. We therefore have

$$\begin{aligned}
-2\mathcal{I}_{+\infty} &= \mathrm{Tr}_{N_c}\,\mathrm{Tr}_{N_f}\left.\frac{(\phi - m_i)}{\sqrt{(\phi - m_i)^2 + q^T(q^T)^\dagger}}\right|_{x=+\infty} \\
&= \sum_{a=1}^{N_c}\sum_{i=1}^{N_f}\frac{(m_{\sigma(a)} - m_i)}{\sqrt{(m_{\sigma(a)} - m_i)^2 + v^2 I_\infty}} \\
&= \sum_{a=1}^{N_c}\left(\sum_{i\notin\Sigma}\mathrm{sign}(m_{\sigma(a)} - m_i) + \sum_{i\in\Sigma}\frac{(m_{\sigma(a)} - m_i)}{\sqrt{(m_{\sigma(a)} - m_i)^2 + v^2}}\right) \\
&= \sum_{a=1}^{N_c}\sum_{i\notin\Sigma}\mathrm{sign}(m_{\sigma(a)} - m_i) + \sum_{a,b=1}^{N_c}\frac{(m_{\sigma(a)} - m_{\sigma(b)})}{\sqrt{(m_{\sigma(a)} - m_{\sigma(b)})^2 + v^2}} \\
&= \sum_{a=1}^{N_c}\sum_{i\notin\Sigma}\mathrm{sign}(m_{\sigma(a)} - m_i)
\end{aligned} \tag{B.21}$$

where in the final equality, we have noticed that the term involving sums over a and b vanishes by anti-symmetry. In fact, this same trick also allows us to rewrite the final term as a sum over all i, rather than the complement of Σ. So, including the contribution from $x = -\infty$ as well, we finally get the following expressions for the index

$$\mathcal{I} = -\frac{1}{2}\sum_{i=1}^{N_f}\sum_{a=1}^{N_c}\left(\mathrm{sign}(m_{\sigma(a)} - m_i) - \mathrm{sign}(m_{\tilde\sigma(a)} - m_i)\right) = -\sum_{a=1}^{N_c}\left(\tilde\sigma(a) - \sigma(a)\right). \tag{B.22}$$

Finally, we want to massage this expression to write it in terms of the root vector $\vec{g}$. We'll rewrite the topological charge as $\vec{g} = (p_1, p_2, \ldots, p_{N_f})$ where $p_k = n_k - n_{k-1} \in \{0, 1\}$ and $\sum_{k=1}^{N_f} p_k = 0$. Since the vacuum set Σ consists of p_1 copies of m_1 and p_2 copies of m_2, we can write $\sum_a (\tilde{\sigma}(a) - \sigma(a)) = \sum_k k p_k$ which, defining $n_0 = n_{N_f} = 0$, allows us to rewrite the index as

$$\mathcal{I} = -\sum_{k=1}^{N_f} k p_k = -\sum_{k=1}^{N_f} k(n_k - n_{k-1}) = \sum_{a=1}^{N_f - 1} n_a \tag{B.23}$$

as advertised.

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References

[1] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[2] L. Alvarez-Gaume and D.Z. Freedman, *Potentials for the supersymmetric nonlinear sigma model*, *Commun. Math. Phys.* **91** (1983) 87.

[3] E.R.C. Abraham and P.K. Townsend, *Q kinks*, *Phys. Lett.* **B 291** (1992) 85; *More on q kinks: a (1+1)-dimensional analog of dyons*, *Phys. Lett.* **B 295** (1992) 225.

[4] J.P. Gauntlett, D. Tong and P.K. Townsend, *Supersymmetric intersecting domain walls in massive hyper-Kaehler sigma models*, *Phys. Rev.* **D 63** (2001) 085001 [hep-th/0007124].

[5] J.P. Gauntlett, D. Tong and P.K. Townsend, *Multi-domain walls in massive supersymmetric sigma-models*, *Phys. Rev.* **D 64** (2001) 025010 [hep-th/0012178].

[6] M. Arai, M. Naganuma, M. Nitta and N. Sakai, *Manifest supersymmetry for BPS walls in $N = 2$ nonlinear sigma models*, *Nucl. Phys.* **B 652** (2003) 35 [hep-th/0211103]; *BPS wall in $N = 2$ SUSY nonlinear sigma model with Eguchi-Hanson manifold*, hep-th/0302028.

[7] A. Losev and M. Shifman, *$N = 2$ sigma model with twisted mass and superpotential: central charges and solitons*, *Phys. Rev.* **D 68** (2003) 045006 [hep-th/0304003].

[8] N.D. Lambert and D. Tong, *Kinky D-strings*, *Nucl. Phys.* **B 569** (2000) 606 [hep-th/9907098].

[9] D. Tong, *The moduli space of BPS domain walls*, *Phys. Rev.* **D 66** (2002) 025013 [hep-th/0202012].

[10] K.S.M. Lee, *An index theorem for domain walls in supersymmetric gauge theories*, *Phys. Rev.* **D 67** (2003) 045009 [hep-th/0211058].

[11] M. Shifman and A. Yung, *Domain walls and flux tubes in $N = 2$ SQCD: D-brane prototypes*, *Phys. Rev.* **D 67** (2003) 125007 [hep-th/0212293].

[12] K. Kakimoto and N. Sakai, *Domain wall junction in $N = 2$ supersymmetric QED in four dimensions*, *Phys. Rev.* **D 68** (2003) 065005 [hep-th/0306077].

[13] Y. Isozumi, K. Ohashi and N. Sakai, *Exact wall solutions in 5-dimensional SUSY QED at finite coupling*, *JHEP* **11** (2003) 060 [hep-th/0310189]; *Massless localized vector field on a wall in $D = 5$ SQED with tensor multiplets*, *JHEP* **11** (2003) 061 [hep-th/0310130].

[14] M. Shifman and A. Yung, *Localization of non-abelian gauge fields on domain walls at weak coupling (D-brane prototypes II)*, *Phys. Rev.* **D 70** (2004) 025013 [hep-th/0312257].

[15] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Construction of non-abelian walls and their complete moduli space*, *Phys. Rev. Lett.* **93** (2004) 161601 [hep-th/0404198].

[16] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Non-abelian walls in supersymmetric gauge theories*, *Phys. Rev.* **D 70** (2004) 125014 [hep-th/0405194].

[17] N. Dorey, *The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*, *JHEP* **11** (1998) 005 [hep-th/9806056];
N. Dorey, T.J. Hollowood and D. Tong, *The BPS spectra of gauge theories in two and four dimensions*, *JHEP* **05** (1999) 006 [hep-th/9902134].

[18] D. Tong, *Mirror mirror on the wall: on two-dimensional black holes and liouville theory*, *JHEP* **04** (2003) 031 [hep-th/0303151].

[19] V.S. Kaplunovsky, J. Sonnenschein and S. Yankielowicz, *Domain walls in supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 552** (1999) 209 [hep-th/9811195].

[20] E.J. Weinberg, *Fundamental monopoles and multi-monopole solutions for arbitrary simple gauge groups*, *Nucl. Phys.* **B 167** (1980) 500.

[21] D. Tong, *Monopoles in the Higgs phase*, *Phys. Rev.* **D 69** (2004) 065003 [hep-th/0307302].

[22] N.D. Lambert and D. Tong, *Kinky D-strings*, *Nucl. Phys.* **B 569** (2000) 606 [hep-th/9907098].

[23] M. Eto et al., *D-brane construction for non-abelian walls*, hep-th/0412024.

[24] N.S. Manton, *A remark on the scattering of BPS monopoles*, *Phys. Lett.* **B 110** (1982) 54.

[25] J.P. Gauntlett, R. Portugues, D. Tong and P.K. Townsend, *D-brane solitons in supersymmetric sigma-models*, *Phys. Rev.* **D 63** (2001) 085002 [hep-th/0008221].

[26] G.R. Dvali and M.A. Shifman, *Domain walls in strongly coupled theories*, *Phys. Lett.* **B 396** (1997) 64 [hep-th/9612128], erratum *ibid.* **B407** (1997) 452.

[27] N. Arkani-Hamed and M. Schmaltz, *Field theoretic branes and tachyons of the QCD string*, *Phys. Lett.* **B 450** (1999) 92 [hep-th/9812010].

[28] A.A. Abrikosov, *On the magnetic properties of superconductors of the second group*, *Sov. Phys. JETP* **5** (1957) 1174;
H.B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, *Nucl. Phys.* **B 61** (1973) 45.

[29] T. Vachaspati and A. Achucarro, *Semilocal cosmic strings*, *Phys. Rev.* **D 44** (1991) 3067;
A. Achucarro and T. Vachaspati, *Semilocal and electroweak strings*, *Phys. Rept.* **327** (2000) 347 [hep-ph/9904229].

JHEP03(2005)019

[30] A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037 [hep-th/0306150].

[31] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nonabelian superconductors: vortices and confinement in $N = 2$ SQCD*, *Nucl. Phys.* **B 673** (2003) 187 [hep-th/0307287].

[32] E.J. Weinberg, *Multivortex solutions of the Ginzburg-Landau equations*, *Phys. Rev.* **D 19** (1979) 3008.

[33] R.S. Ward, *Slowly moving lumps in the CP^1 model in (2+1)-dimensions*, *Phys. Lett.* **B 158** (1985) 424;
R.A. Leese and T.M. Samols, *Interaction of semilocal vortices*, *Nucl. Phys.* **B 396** (1993) 639.

[34] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *All exact solutions of a 1/4 Bogomol'nyi-prasad-sommerfield equation*, hep-th/0405129.

[35] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Instantons in the Higgs phase*, hep-th/0412048.

[36] M. Shifman and A. Yung, *Non-abelian string junctions as confined monopoles*, *Phys. Rev.* **D 70** (2004) 045004 [hep-th/0403149].

[37] A. Hanany and D. Tong, *Vortex strings and four-dimensional gauge dynamics*, *JHEP* **04** (2004) 066 [hep-th/0403158].

[38] A. Hanany and K. Hori, *Branes and $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 513** (1998) 119 [hep-th/9707192].

[39] M.A.C. Kneipp, *Z(k) string fluxes and monopole confinement in non-abelian theories*, *Phys. Rev.* **D 68** (2003) 045009 [hep-th/0211049];
R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, *Nonabelian monopoles and the vortices that confine them*, *Nucl. Phys.* **B 686** (2004) 119 [hep-th/0312233];
V. Markov, A. Marshakov and A. Yung, *Non-abelian vortices in $N = 1*$ gauge theory*, *Nucl. Phys.* **B 709** (2005) 267 [hep-th/0408235];
R. Auzzi, S. Bolognesi and J. Evslin, *Monopoles can be confined by 0, 1 or 2 vortices*, *JHEP* **02** (2005) 046 [hep-th/0411074];
A. Gorsky, M. Shifman and A. Yung, *Non-abelian meissner effect in Yang-Mills theories at weak coupling*, *Phys. Rev.* **D 71** (2005) 045010 [hep-th/0412082].

[40] P. Goddard, J. Nuyts and D.I. Olive, *Gauge theories and magnetic charge*, *Nucl. Phys.* **B 125** (1977) 1.

[41] S.M. Carroll and M. Trodden, *Dirichlet topological defects*, *Phys. Rev.* **D 57** (1998) 5189 [hep-th/9711099];
G. Dvali and A. Vilenkin, *Solitonic D-branes and brane annihilation*, *Phys. Rev.* **D 67** (2003) 046002 [hep-th/0209217];
M. Bowick, A. De Felice and M. Trodden, *The shapes of Dirichlet defects*, *JHEP* **10** (2003) 067 [hep-th/0306224].

[42] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658 [hep-th/9706109];
A. Hanany, M.J. Strassler and A. Zaffaroni, *Confinement and strings in MQCD*, *Nucl. Phys.* **B 513** (1998) 87 [hep-th/9707244];
A. Campos, K. Holland and U.J. Wiese, *Complete wetting in supersymmetric QCD or why QCD strings can end on domain walls*, *Phys. Rev. Lett.* **81** (1998) 2420 [hep-th/9805086].

[43] N.D. Mermin, *Surface singularities and superflow in ^{3}He-A*, in *Quantum fluids and solids*, S.B. Trickey, E.D. Adams and J.W. Dufty eds., Plenum Press, New York 1977.

[44] G.E. Volovik, *Defects at interface between A and B phases of superfluid ^{3}He*, *JETP Lett.* **51** (1990) 449.

[45] G.E. Volovik, *The universe in a helium droplet*, Oxford University Press, Oxford 2003.

[46] D.L. Stein, R.D. Pisarski, and P.W. Anderson, *Boojums in superfluid ^{3}He-A and cholesteric liquid crystals*, *Phys. Rev. Lett.* **40** (1978) 1269;
S.A. Langer and J.P. Sethna, *Textures in a chiral smectic liquid-crystal film*, *Phys. Rev.* **A 34** (1986) 5035.

[47] G.W. Gibbons and P.K. Townsend, *A bogomolnyi equation for intersecting domain walls*, *Phys. Rev. Lett.* **83** (1999) 1727 [hep-th/9905196];
S.M. Carroll, S. Hellerman and M. Trodden, *Domain wall junctions are 1/4-BPS states*, *Phys. Rev.* **D 61** (2000) 065001 [hep-th/9905217].

[48] H. Oda, K. Ito, M. Naganuma and N. Sakai, *An exact solution of BPS domain wall junction*, *Phys. Lett.* **B 471** (1999) 140 [hep-th/9910095];
K. Ito, M. Naganuma, H. Oda and N. Sakai, *Nonnormalizable zero modes on BPS junctions*, *Nucl. Phys.* **B 586** (2000) 231 [hep-th/0004188]; *An exact solution of BPS junctions and its properties*, *Nucl. Phys.* **101** *(Proc. Suppl.)* (2001) 304 [hep-th/0012182].

[49] C.G. Callan Jr. and J.M. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147];
G.W. Gibbons, *Born-Infeld particles and Dirichlet p-branes*, *Nucl. Phys.* **B 514** (1998) 603 [hep-th/9709027].